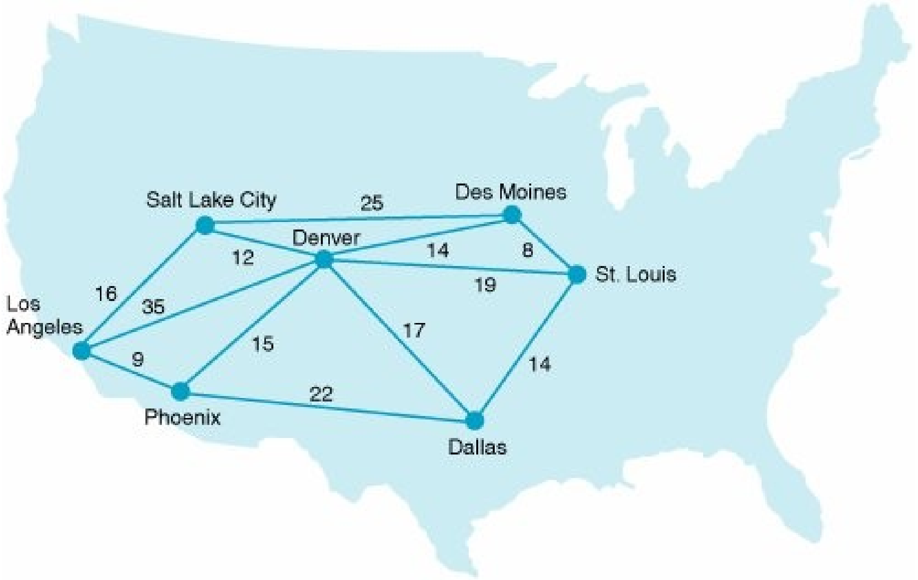
The **shortest route problem** is to determine the shortest distance between an originating point and several destination points. For example, the Stagecoach Shipping Company transports oranges by six trucks from Los Angeles to six cities in the West and Midwest. The different routes between Los Angeles and the destination cities and the length of time, in hours, required by a truck to travel each route are shown in Figure 7.2.

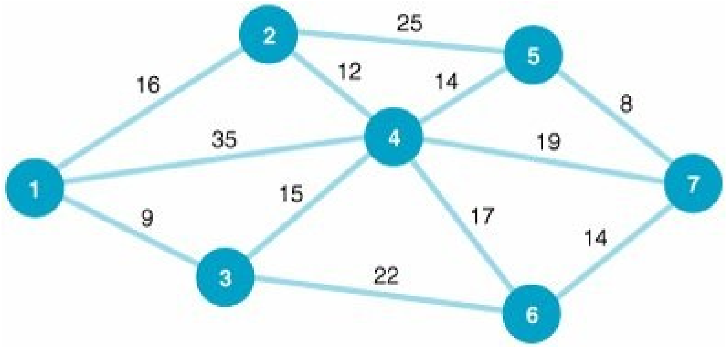
Figure 7.2. Shipping routes from Los Angeles



The shortest route problem is to find the shortest distance between an origin and various destination points .

The shipping company manager wants to determine the best routes (in terms of the minimum travel time) for the trucks to take to reach their destinations. This problem can be solved by using the shortest route solution technique. In applying this technique, it is convenient to represent the system of truck routes as a network, as shown in Figure 7.3.

Figure 7.3. Network of shipping routes

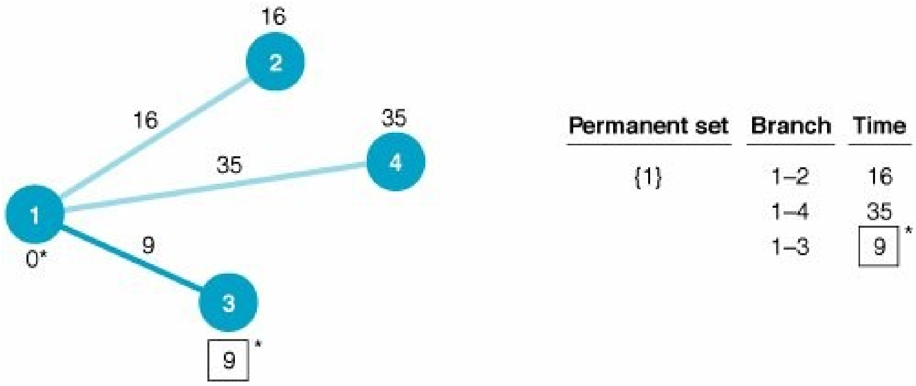


To repeat our objective as it relates to Figure 7.3, we want to determine the shortest routes from the origin (node 1) to the six destinations (nodes 2 through 7).

The Shortest Route Solution Approach

We begin the shortest route solution technique by starting at node 1 (the origin) and determining the shortest time required to get to a directly connected (i.e., adjacent) node. The three nodes directly connected to node 1 are 2, 3, and 4, as shown in Figure 7.4. Of these three nodes, the shortest time is 9 hours to node 3. Thus, we have determined our first shortest route from nodes 1 to 3 (i.e., from Los Angeles to Phoenix). We will now refer to nodes 1 and 3 as the **permanent set** to indicate that we have found the shortest route to these nodes. (Because node 1 has no route to it, it is automatically in the permanent set.)

Figure 7.4. Network with node 1 in the permanent set



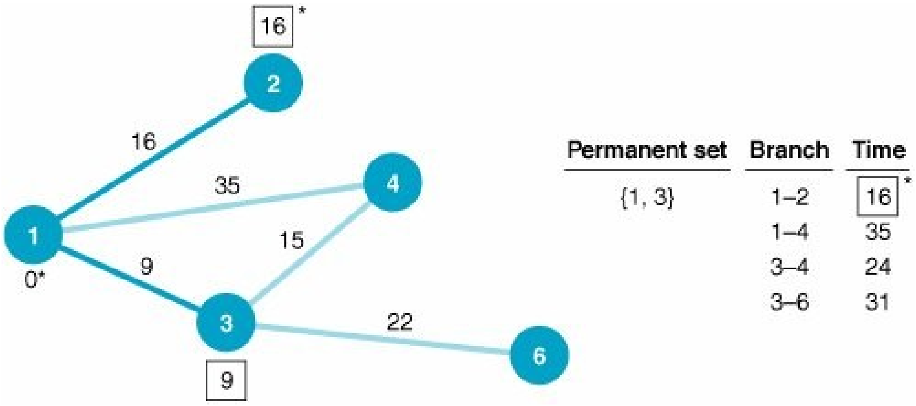
Permanent set	Branch	Time
{1}	1–2	16
	1–4	35
	1–3	$\boxed{9}$ *

Determine the initial shortest route from the origin (node 1) to the closest node (3) .

Notice in Figure 7.4 that the shortest route to node 3 is drawn with a heavy line, and the shortest time to node 3 (9 hours) is enclosed by a box. The table accompanying Figure 7.4 describes the process of selecting the shortest route. The permanent set is shown to contain only node 1. The three branches from node 1 are 12, 14, and 13, and this last branch has the minimum time of 9 hours.

Next , we will repeat the foregoing steps used to determine the shortest route to node 3. First, we must determine all the nodes directly connected to the nodes in the permanent set (nodes 1 and 3). Nodes 2, 4, and 6 are all directly connected to nodes 1 and 3, as shown in Figure 7.5.

Figure 7.5. Network with nodes 1 and 3 in the permanent set



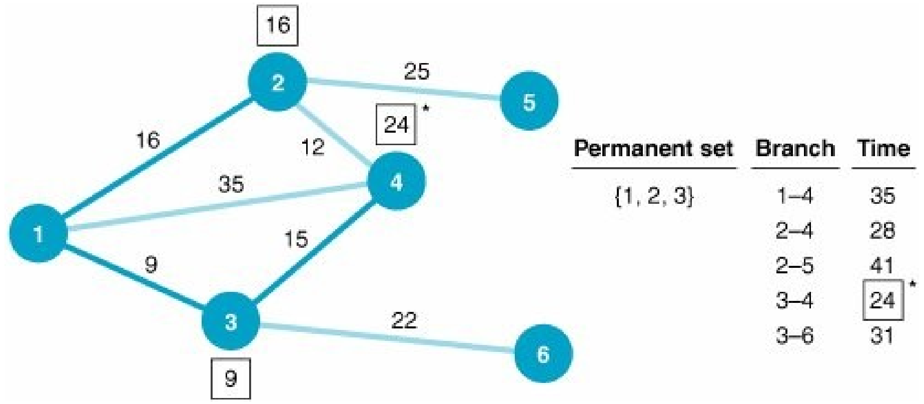
Permanent set	Branch	Time
{1, 3}	1–2	$\boxed{16}$ *
	1–4	35
	3–4	24
	3–6	31

Determine all nodes directly connected to the permanent set .

The next step is to determine the shortest route to the three nodes (2, 4, and 6) directly connected to the permanent set nodes. There are two branches starting from node 1 (12 and 14) and two branches from node 3 (34 and 36). The branch with the shortest time is to node 2, with a time of 16 hours. Thus, node 2 becomes part of the permanent set. Notice in our computations accompanying Figure 7.5 that the time to node 6 (branch 36) is 31 hours, which is determined by adding the branch 36 time of 22 hours to the shortest route time of 9 hours at node 3.

As we move to the next step, the permanent set consists of nodes 1, 2, and 3. This indicates that we have found the shortest route to nodes 1, 2, and 3. We must now determine which nodes are directly connected to the permanent set nodes. Node 5 is the only adjacent node not presently connected to the permanent set, so it is connected directly to node 2. In addition, node 4 is now connected directly to node 2 (because node 2 has joined the permanent set). These additions are shown in Figure 7.6.

Figure 7.6. Network with nodes 1, 2, and 3 in the permanent set



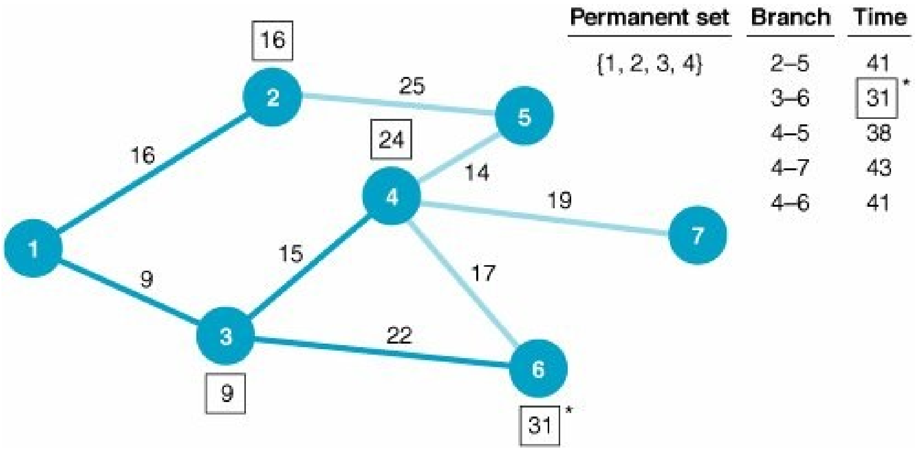
Permanent set	Branch	Time
{1, 2, 3}	1–4	35
	2–4	28
	2–5	41
	3–4	$\boxed{24}$*
	3–6	31

Five branches lead from the permanent set nodes (1, 2, and 3) to their directly connected nodes, as shown in the table accompanying Figure 7.6. The branch representing the route with the shortest time is 34, with a time of 24 hours. Thus, we have determined the shortest route to node 4, and it joins the permanent set. Notice that the shortest time to node 4 (24 hours) is the route from node 1 through node 3. The other routes to node 4 from node 1 through node 2 are longer; therefore, we will not consider them any further as possible routes to node 4.

To summarize, the shortest routes to nodes 1, 2, 3, and 4 have all been determined, and these nodes now form the permanent set.

Next, we repeat the process of determining the nodes directly connected to the permanent set nodes. These directly connected nodes are 5, 6, and 7, as shown in Figure 7.7. Notice in Figure 7.7 that we have eliminated the branches from nodes 1 and 2 to node 4 because we determined that the route with the shortest time to node 4 does not include these branches.

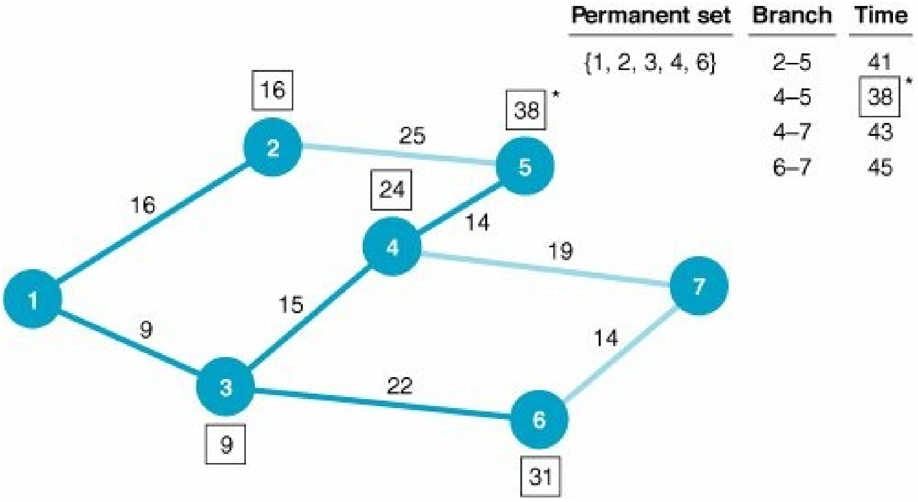
Figure 7.7. Network with nodes 1, 2, 3, and 4 in the permanent set



Permanent set	Branch	Time
{1, 2, 3, 4}	2–5	41
	3–6	31*
	4–5	38
	4–7	43
	4–6	41

From the table accompanying Figure 7.7 we can see that of the branches leading to nodes 5, 6, and 7, branch 36 has the shortest cumulative time, of 31 hours. Thus, node 6 is added to our permanent set. This means that we have now found the shortest routes to nodes 1, 2, 3, 4, and 6.
Repeating the process, we observe that the nodes directly connected (adjacent) to our permanent set are nodes 5 and 7, as shown in Figure 7.8. (Notice that branch 46 has been eliminated because the best route to node 6 goes through node 3 instead of node 4.)

Figure 7.8. Network with nodes 1, 2, 3, 4, and 6 in the permanent set

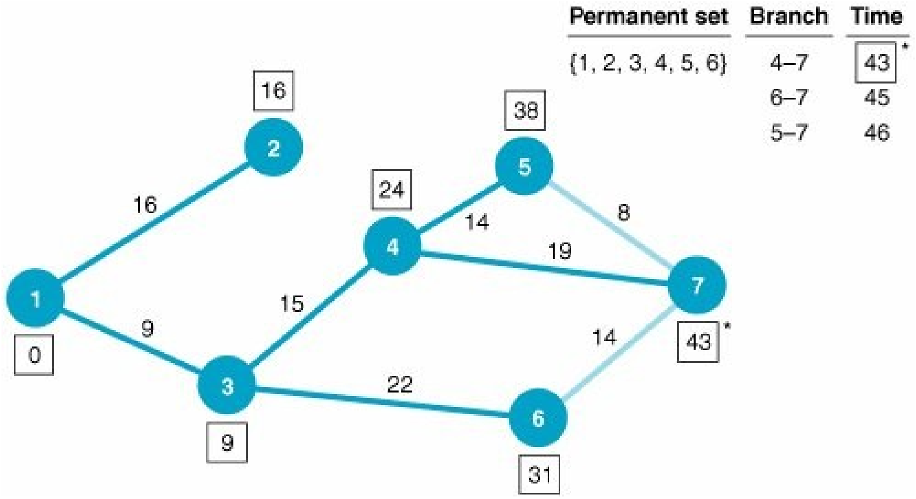


Permanent set	Branch	Time
{1, 2, 3, 4, 6}	2–5	41
	4–5	38*
	4–7	43
	6–7	45

Of the branches leading from the permanent set nodes to nodes 5 and 7, branch 45 has the shortest cumulative time, of 38 hours. Thus, node 5 joins the permanent set. We have now determined the routes with the shortest times to nodes 1, 2, 3, 4, 5, and 6 (as denoted by the heavy branches in Figure 7.8).

The only remaining node directly connected to the permanent set is node 7, as shown in Figure 7.9. Of the three branches connecting node 7 to the permanent set, branch 47 has the shortest time, of 43 hours. Therefore, node 7 joins the permanent set.

Figure 7.9. Network with nodes 1, 2, 3, 4, 5, and 6 in the permanent set



Permanent set	Branch	Time
{1, 2, 3, 4, 5, 6}	4–7	43*
	6–7	45
	5–7	46

The routes with the shortest times from the origin (node 1) to each of the other six nodes and their corresponding travel times are summarized in Figure 7.10 and Table 7.1.

Figure 7.10. Network with optimal routes from Los Angeles to all destinations

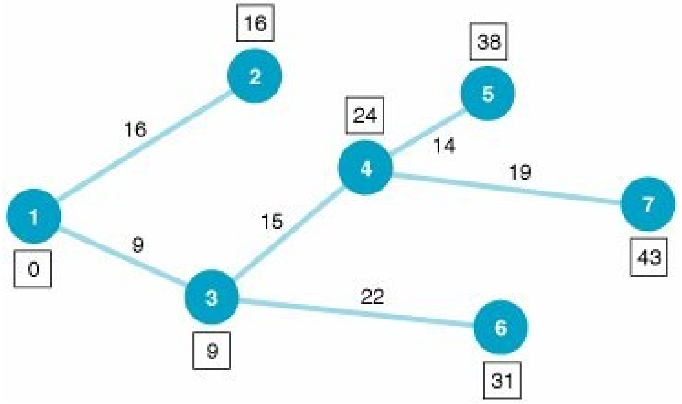


Table 7.1. Shortest travel time from origin to each destination

Salt Lake City (node 2)	12	16
Phoenix (node 3)	13	9
Denver (node 4)	134	24
Des Moines (node 5)	1345	38
Dallas (node 6)	136	31
St. Louis (node 7)	1347	43

In summary, the steps of the shortest route solution method are as follows :

1. Select the node with the shortest direct route from the origin.
2. Establish a permanent set with the origin node and the node that was selected in step 1.
3. Determine all nodes directly connected to the permanent set nodes.
4. Select the node with the shortest route (branch) from the group of nodes directly connected to the permanent set nodes.
5. Repeat steps 3 and 4 until all nodes have joined the permanent set.

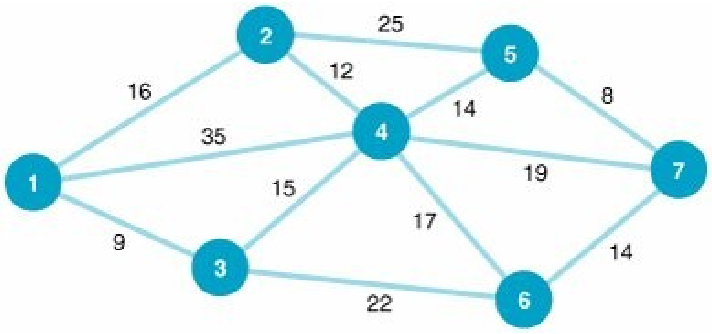
The Minimal Spanning Tree Problem

In the shortest route problem presented in the previous section, the objective was to determine the shortest routes between the origin and the destination nodes in the network. In our example, we determined the best route from Los Angeles to each of the six destination cities. The **minimal spanning tree problem** is similar to the shortest route problem, except that the objective is to connect all the nodes in the network so that the total branch lengths are minimized. The resulting network spans (connects) all the points in the network at a minimum total distance (or length).

The minimal spanning tree problem is to connect all nodes in a network so that the total branch lengths are minimized

To demonstrate the minimal spanning tree problem, we will consider the following example. The Metro Cable Television Company is to install a television cable system in a community consisting of seven suburbs. Each of the suburbs must be connected to the main cable system. The cable television company wants to lay out the main cable network in a way that will minimize the total length of cable that must be installed. The possible paths available to the cable television company (by consent of the town council) and the feet of cable (in thousands of feet) required for each path are shown in Figure 7.11.

Figure 7.11. Network of possible cable TV paths



In Figure 7.11 the branch from node 1 to node 2 represents the available cable path between suburbs 1 and 2. The branch requires 16,000 feet of cable. Notice that the network shown in Figure 7.11 is identical to the network in Figure 7.2 that we used to demonstrate the shortest route problem. The networks were intentionally made identical to demonstrate the difference between the results of the two types of network models.

The Minimal Spanning Tree Solution Approach

The solution approach to the minimal spanning tree problem is actually easier than the shortest route solution method. In the minimal spanning tree solution approach, we can start at any node in the network. However, the conventional approach is to start with node 1. Beginning at node 1, we select the closest node (i.e., the shortest branch) to join our spanning tree. The shortest branch from node 1 is to node 3, with a length of 9 (thousand feet). This branch is indicated with a heavy line in Figure 7.12. Now we have a spanning tree consisting of two nodes: 1 and 3. The next step is to select the closest node not now in the spanning tree. The node closest to either node 1 or node 3 (the nodes in our present spanning tree) is node 4, with a branch length of 15,000 feet. The addition of node 4 to our spanning tree is shown in Figure 7.13.

Figure 7.12. Spanning tree with nodes 1 and 3

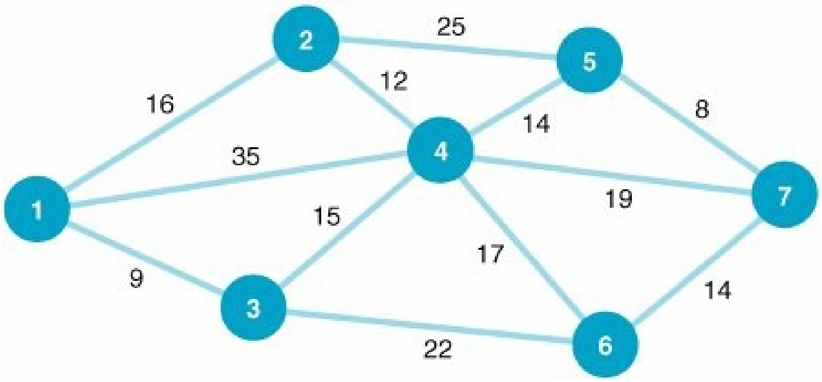
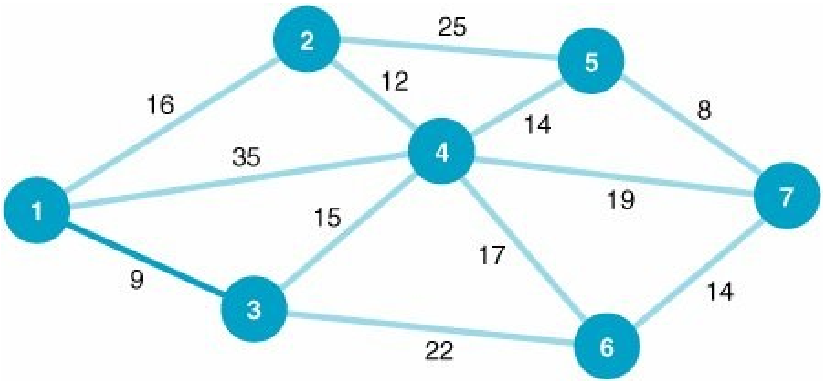


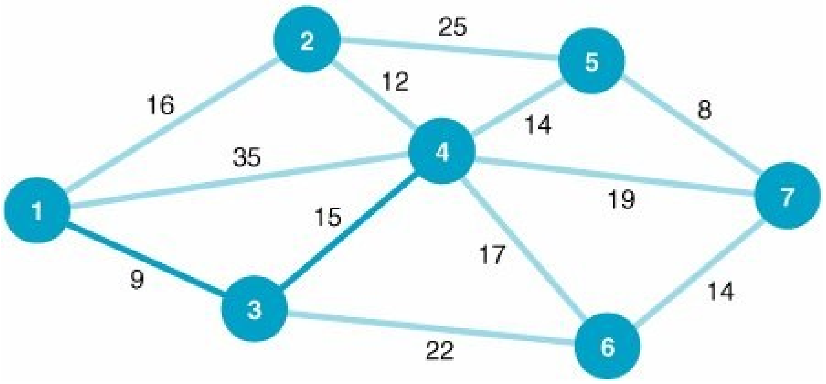
Figure 7.13. Spanning tree with nodes 1, 3, and 4



Start with any node in the network and select the closest node to join the spanning tree. Select the closest node to any node in the spanning area .

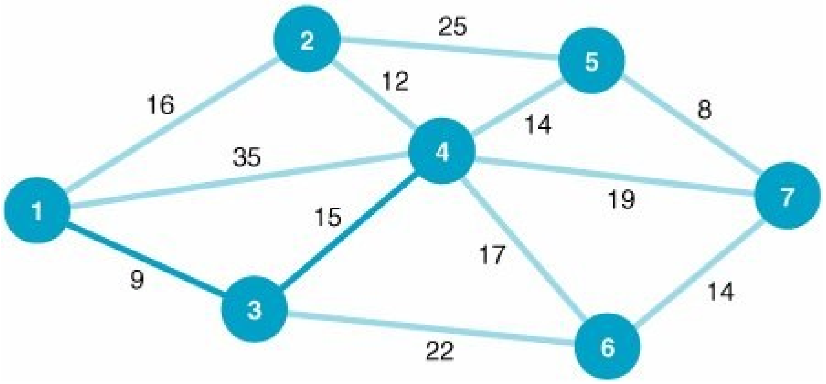
Next, we repeat the process of selecting the closest node to our present spanning tree (nodes 1, 3, and 4). The closest node not now connected to the nodes in our spanning tree is node 2. The length of the branch from node 4 to node 2 is 12,000 feet. The addition of node 2 to the spanning tree is shown in Figure 7.14.

Figure 7.14. Spanning tree with nodes 1, 2, 3, and 4



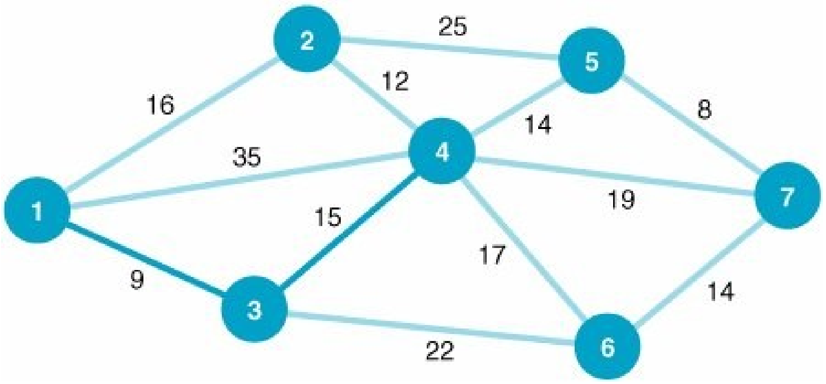
Our spanning tree now consists of nodes 1, 2, 3, and 4. The node closest to this spanning tree is node 5, with a branch length of 14,000 feet to node 4. Thus, node 5 joins our spanning tree, as shown in Figure 7.15.

Figure 7.15. Spanning tree with nodes 1, 2, 3, 4, and 5



The spanning tree now contains nodes 1, 2, 3, 4, and 5. The closest node not currently connected to the spanning tree is node 7. The branch connecting node 7 to node 5 has a length of 8,000 feet. Figure 7.16 shows the addition of node 7 to the spanning tree.

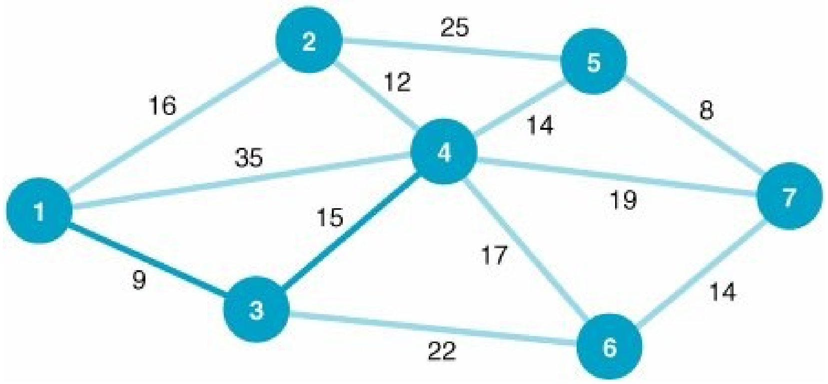
Figure 7.16. Spanning tree with nodes 1, 2, 3, 4, 5, and 7



Now our spanning tree includes nodes 1, 2, 3, 4, 5, and 7. The only remaining node not connected to the spanning tree is node 6. The node in the spanning tree closest to node 6 is node 7, with a branch length of 14,000 feet. The complete spanning tree, which now includes all seven nodes, is shown in Figure 7.17.

Figure 7.17. Minimal spanning tree for cable TV network

(This item is displayed on page 285 in the print version)



The spanning tree shown in Figure 7.17 requires the minimum amount of television cable to connect the seven suburbs72,000 feet. This same minimal spanning tree could have been obtained by starting at any of the six nodes other than node 1.

Notice the difference between the minimal spanning tree network shown in Figure 7.17 and the shortest route network shown in Figure 7.10. The shortest route network represents the shortest paths between the origin and each of the destination nodes (i.e., six different routes). In contrast, the minimal spanning tree network shows how to connect all seven nodes so that the total distance (length) is minimized.

In summary, the steps of the minimal spanning tree solution method are as follows :

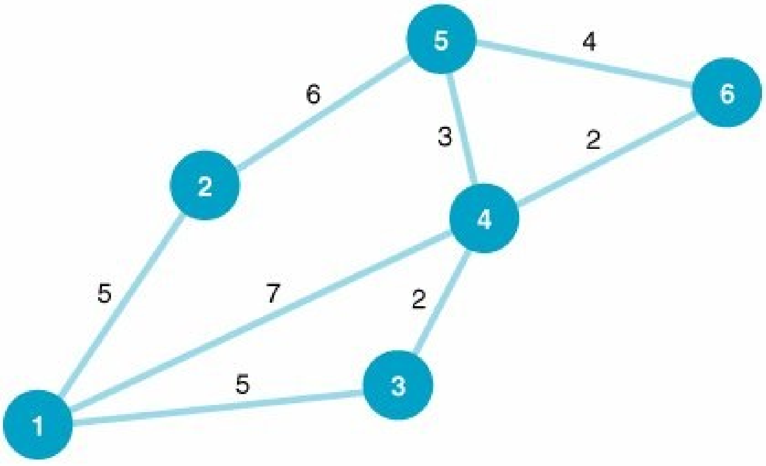
1. Select any starting node (conventionally, node 1 is selected).
2. Select the node closest to the starting node to join the spanning tree.
3. Select the closest node not presently in the spanning tree.
4. Repeat step 3 until all nodes have joined the spanning tree.

Example Problem Solution

The following example illustrates the solution methods for the shortest route and minimal spanning tree network flow problems.

Problem Statement

A salesman for Healthproof Pharmaceutical Company travels each week from his office in Atlanta to one of five cities in the Southeast where he has clients . The travel time (in hours) between cities along interstate highways is shown along each branch in the following network:

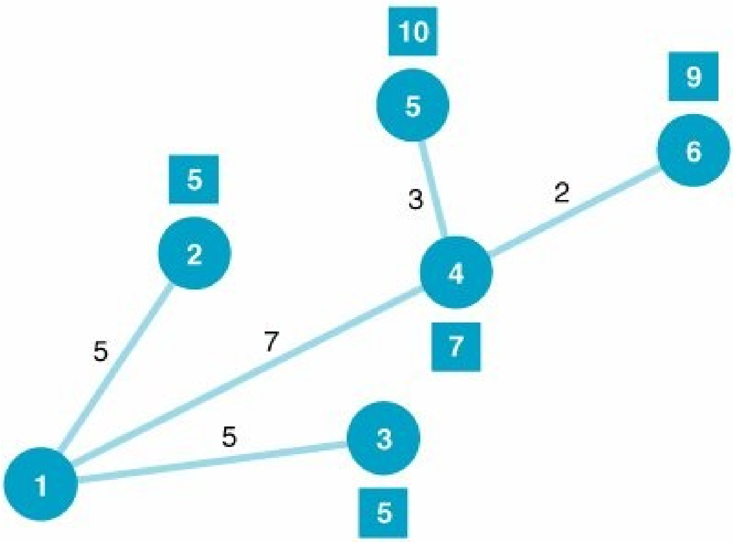


a. Determine the shortest route from Atlanta to each of the other five cities in the network.

b. Assume that the network now represents six different communities in a city and that the local transportation authority wants to design a rail system that will connect all six communities with the minimum amount of track. The miles between each community are shown on each branch. Develop a minimal spanning tree for this problem.

Solution

Step 1. (Part A): Determine the Shortest Route Solution



Step 2. (Part B): Determine the Minimal Spanning Tree

The minimal spanning tree follows; the shortest total distance is 17 miles: