Chapter 4. The simplex algorithm

- Putting Linear Programs into standard form
- Introduction to Simplex Algorithm

Overview of Lecture

- Getting an LP into standard form
- Getting an LP into canonical form
- Optimality conditions
- Improving a solution
- A simplex pivot
- Recognizing when an LP is unbounded
- Overview of what remains

Overview of Lecture

- Getting an LP into standard form

Overview of Lecture

- **Getting an LP into standard form**

Linear Programs in Standard Form

We say that a linear program is ***in standard form*** if the following are all true:

1. Non-negativity constraints for all variables.
2. All remaining constraints are expressed as equality constraints (b/c we'll need to solve systems of linear equations)
3. The right hand side vector, *b,* is non-negative.

An LP not in Standard Form

maximize $3x_1 + 2x_2 - x_3 + x_4$

$x_1 + 2x_2 + x_3 - x_4 \leq 5;$ not equality

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1;$ not equality

$x_1 \geq 0, x_2 \geq 0$ x_3 may be negative

Converting Inequalities into Equalities Plus Non-negatives

Before

$x_1 + 2x_2 + x_3 - x_4 \leq 5$

After

$x_1 + 2x_2 + x_3 - x_4 + s_1 = 5$

$s_1 \geq 0$

s_1 is called a *slack variable*, which measures the amount of "unused resource."

Note that $s_1 = 5 - x_1 - 2x_2 - x_3 + x_4$.

To convert a "≤" constraint to an equality, add a slack variable.

Converting "≥" constraints

- Consider the inequality $-2x_1 - 4x_2 + x_3 + x_4 \leq -1$;
- Step 1. Eliminate the negative RHS

$$2x_1 + 4x_2 - x_3 - x_4 \geq 1$$

- Step 2. Convert to an equality

$$2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1$$

$$s_2 \geq 0$$

- The variable added will be called a "***surplus variable.***"

To convert a "≥" constraint to an equality, subtract a surplus variable.

More Transformations

How can one convert a maximization problem to a minimization problem?

Example: Maximize 3W + 2P

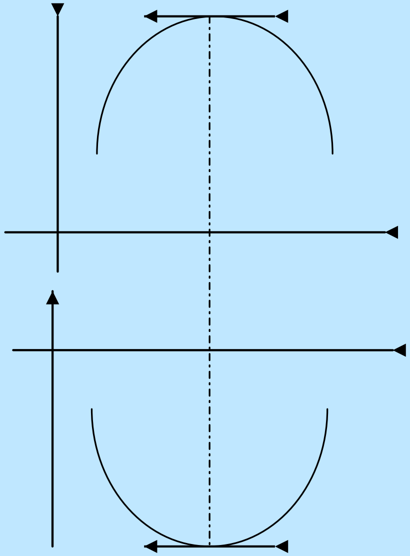
Subject to "constraints"

Has the same optimum solution(s) as

Minimize -3W -2P

Subject to "constraints"

Max f(x) vs Min -f(x): don’t forget the “-”



The maximum of f(x) is the negative of the minimum of – f(x), and the min and the max are reached for the same value(s) of x.

The Last Transformations (for now)

Transforming variables that may take on negative values.

maximize $3x_1 + 4x_2 + 5\ x_3$

subject to $2x_1\ - 5x_2\ + 2x_3 \quad = \quad 7$

other constraints

$x_1 \leq\ 0$, x_2 is unconstrained in sign, $x_3 \geq 0$

Transforming x_1: replace x_1 by $y_1 = -x_1;\ \ y_1 \geq 0.$

max $-3\ y_1 + 4x_2 + 5\ x_3$

$-2\ y_1 - 5\ x_2 + 2\ x_3 = 7$

$y_1 \geq\ \ 0$, x_2 is unconstrained in sign, $x_3 \geq 0$

One can recover x_1 from y_1.

Transforming variables that may take on positive, zero or negative values.

$$\max\ -3y_1 + 4x_2 + 5x_3$$

$$-2y_1 - 5x_2 + 2x_3 = 7$$

$y_1 \geq 0$, x_2 is unconstrained in sign, $x_3 \geq 0$

e.g., $y_1 = 1$, $x_2 = -1$, $x_3 = 2$ is feasible.

Transforming x_2: replace x_2 by $x_2 = y_3 - y_2$; $y_2 \geq 0$, $y_3 \geq 0$.

$$\max\ -3y_1 + 4(y_3 - y_2) + 5x_3$$

$$-2y_1 - 5(y_3 - y_2) + 2x_3 = 7$$

$$\text{all vars} \geq 0$$

One can recover x_2 from y_2, y_3.

e.g., $y_1 = 1$, $y_2 = 0$, $y_3 = 1$, $x_3 = 2$ is feasible.

Variables that are unrestricted in sign

- Take y that can be any sign. If we write

  y = y1 – y2, y1 and y2 $\geq$ 0,

then if y = 2, one could choose y1=2 and y2=0,

or y1=2+a, y2=a, for any a $\geq$ 0.

and if y = -3, one could similarly obtain

y1=0 and y2=3,

or y1=a and y2=a-3, for any a $\geq$ 0.

In fact, in the optimal solution of an LP obtained by the simplex method, one of the two y1 or y2 will always be 0. So only a blue solution can be found, and y1 = y+ = max{0,y} and y2 = y- = max{0, -y}.

Another Example

- Exercise: transform the following to standard form (maximization):

Minimize $x_1 + 3x_2$

Subject to $2x_1 + 5x_2 \leq 12$

$x_1 + x_2 \geq 1$

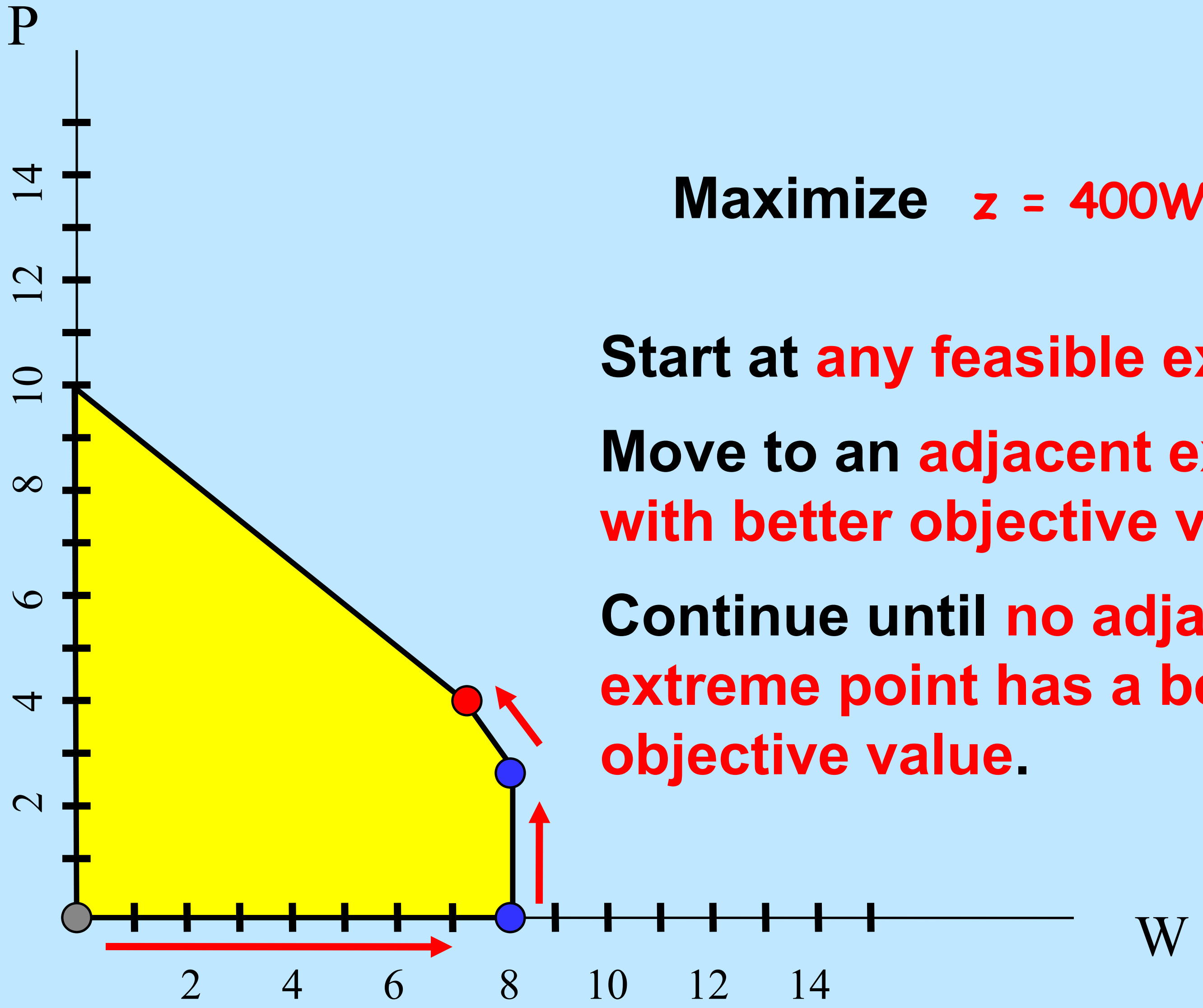
$x_1 \geq 0$

Perform the transformation with your partner

The Algorithm???

- We will study the simplex method due to George Dantzig (~1947).
- Another type of approach was suggested by Dinic and Huard in particular in the 60’s, they are “interior point methods”. The method proposed by Karmarkar made headlines in 1985.
- Both approaches co-exist, and most modern commercial programs offer both.

Remember the Simplex Method?



Maximize $z = 400W + 300P$

Start at any feasible extreme point.

Move to an adjacent extreme point with better objective value.

Continue until no adjacent extreme point has a better objective value.

Preview of the Simplex Algorithm

maximize $z = -3x_1 + 2x_2$

subject to

$$\begin{array}{rrrrcl} -3x_1 & + 3x_2 & + x_3 & & = & 6 \\ -4x_1 & + 2x_2 & & + x_4 & = & 2 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

-z	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Overview of Lecture

- Getting an LP into standard form
- Getting an LP into canonical form

LP Canonical Form =
LP Standard Form + Jordan Canonical Form

$-z$	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

z is not a decision variable

The simplex method starts with an LP in LP canonical form (or it creates canonical form at a preprocess step.)

LP Canonical Form

$-z$	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

z is not a decision variable

The basic variables are x_3 and x_4.

The non-basic variables are x_1 and x_2.

The basic feasible solution is $x_1 = 0, x_2 = 0, x_3 = 6, x_4 = 2$

(set the non-basic variables to 0, and then solve)

For each constraint there is a basic variable

-z	x_1	x_2	x_3	x_4			
1	-3	2	0	0	=	0	
0	-3	3	1	0	=	6	Constraint 1
0	-4	2	0	1	=	2	Constraint 2

Constraint 1: basic variable is x_3

Constraint 2: basic variable is x_4

The basis consists of variables x_3 and x_4

Overview of Lecture

- Getting an LP into standard form
- Getting an LP into canonical form
- Optimality conditions

z	x_1	x_2	x_3	x_4		
-1	3	-2	0	0	=	0

Note:
$z = -3x_1 + 2x_2$

-z	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Obvious Fact: If one can improve the current basic feasible solution x, then x is not optimal.

Idea: assign a small value to just one of the non-basic variables, and then adjust the basic variables.

The current basic feasible solution (bfs) is not optimal!

-z	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

If there is a positive coefficient in the -z row, the basis is not optimal**

Recall: $z = -3x_1 + 2x_2$

Increase x_2 to $\Delta > 0$. Let x_1 stay at 0.

What happens to x_3, x_4 and z? .

$x_3 = 6 - 3\Delta.$

$x_4 = 2 - 2\Delta.$

$z = 2\Delta.$

Optimality Conditions
(note that the data is different here)

-z	x_1	x_2	x_3	x_4		
1	-2	-4	0	0	=	-8
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Important Fact. If there is no positive coefficient in the -z row, the basic feasible solution is optimal!

$z = -2x_1 - 4x_2 + 8.$

Therefore $z \leq 8$ for all feasible solutions.

But $z = 8$ in the current basic feasible solution

This basic feasible solution is optimal!

Let $x_2 = \Delta$. How large can Δ be?

What is the solution after changing x_2?

-z	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

$x_1 = 0$

$x_2 = 1$

$x_3 = 3$

$x_4 = 0$

$z = 2$

What is the value of Δ that maximizes z, but leaves a feasible solution?

$\Delta = 1.$

Fact. The resulting solution is a basic feasible solution for a different basis.

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- Getting an LP into standard form
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- Improving a solution, a pivot

Pivoting to obtain a better solution

New Solution: basic variables are x_2 and x_3.

Nonbasics: x_1 and x_4.

$-z$	x_1	x_2	x_3	x_4		
1	1	0	0	-1	=	-2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$x_1 = 0$

$x_2 = 1$

$x_3 = 3$

$x_4 = 0$

$z = 2$

If we pivot on the coefficient 2, we obtain the new basic feasible solution.

Stopped here on
Tuesday 17 September

Summary of Simplex Algorithm

- Start in (Jordan) canonical form with a basic feasible solution

1. Check for optimality conditions
2. If not optimal, determine a non-basic variable that should be made positive
3. Increase that non-basic variable, and perform a pivot, obtaining a new bfs
4. Continue until optimal (or unbounded).

OK. Let’s iterate again.

$z = x_1 - x_4 + 2$

-z	x_1	x_2	x_3	x_4		
1	1	0	0	-1	=	-2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$x_1 = \Delta$

$x_2 = 1 + 2\Delta$

$x_3 = 3 - 3\Delta.$

$x_4 = 0$

$z = 2 + \Delta$

The cost coefficient of x_1 is positive.

Set $x_1 = \Delta$ and keep $x_4 = 0$.

How large can Δ be?

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A Digression: What if we had a problem in which Δ could increase to infinity? Example:

-z	x_1	x_2	x_3	x_4		
1	1	0	0	-1	=	-2
0	-3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$z = x_1 - x_4 + 2$

$x_1 = \Delta$

$x_2 = 1 + 2\Delta$

$x_3 = 3 + 3\Delta.$

$x_4 = 0$

$z = 2 + \Delta$

Suppose we change the 3 to a –3.

Set $x_1 = \Delta$ and keep $x_4 = 0$.

How large can Δ be?

If the non-cost coefficients in the entering column are ≤ 0, then the solution is ***unbounded***

End Digression: Perform another pivot

-z	x_1	x_2	x_3	x_4		
1	0	0	-1/3	-1/2	=	-3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	-1/2	=	3

$x_1 = 1$
$x_2 = 3$
$x_3 = 0$
$x_4 = 0$
$z = 3$

What is the largest value of Δ ? $\Delta = 1$
Variable x_1 becomes basis, x_3 becomes nonbasic.
So, x_1 becomes the basic variable for constraint 1.
Pivot on the coefficient with a 3.

Check for optimality

$$z = -x_3/3 - x_4/2 + 3$$

-z	x_1	x_2	x_3	x_4		
1	0	0	-1/3	-1/2	=	-3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	1/2	=	3

$x_1 = 1$
$x_2 = 3$
$x_3 = 0$
$x_4 = 0$
$z = 3$

There is no positive coefficient in the z-row.

The current basic feasible solution is optimal!

Summary of Simplex Algorithm Again

- Start in (Jordan) canonical form with a basic feasible solution

1. Check for optimality conditions
   - Is there a positive coefficient in the cost row?
2. If not optimal, determine a non-basic variable that should be made positive
   - Choose a variable with a positive coef. in the cost row.
3. increase that non-basic variable, and perform a pivot, obtaining a new bfs (or unboundedness)
   - We will review this step, and show a shortcut
4. Continue until optimal (or unbounded).

Performing a "Pivot". Towards a shortcut.

$z = 2x_1 + 3$

-z	x_1	x_2	x_3	x_4		
1	2	0	0	0	=	-3
0	3	0	1	0	=	7
0	-2	1	0	0	=	1
0	2	0	0	1	=	5

Exercise: to do with your partner.

$x_1 = \Delta$

$x_2 = 1 + 2\Delta$

$x_3 = 7 - 3\Delta.$

$x_4 = 5 - 2\Delta$

$z = 3 + 2\Delta$

1. Determine how large Δ can be.
2. Determine the next solution.
3. Determine what coefficient should be pivoted on.
4. The choice depends on ratios involving the coefficients. What is the rule for determining the coef?

More on performing a pivot

- To determine the column to pivot on, select a variable with a positive cost coefficient
- To determine a row to pivot on, select a coefficient according to a minimum ratio rule
- Carry out a pivot as one does in solving a system of equations.

One more modeling rule: whenever possible, one wants a Linear Program

- Suppose x is the total number of chairs produced
- Suppose y is the number of black chairs
- At least 30% of chairs must be black:

  $y/x \geq 0.30$ (assumes x nonzero)

  This is not linear!!!!!

Always transform into a linear constraint:

$y - 0.30x \geq 0$ (works even for $x=0$!)

Next Lecture

- Review of the simplex algorithm
- Formalizing the simplex algorithm
- How to find an initial basic feasible solution, if one exists
- A proof that the simplex algorithm is finite (assuming non-degeneracy)