Simplex Method Continued

Next Lectures

- Review of the simplex algorithm.
- Formalizing the approach
- Degeneracy and Alternative Optimal Solutions
- Is the simplex algorithm finite? (Answer, yes, but only if we are careful)

LP Canonical Form = LP Standard Form + Jordan Canonical Form

BV	$-z$	x_1	x_2	x_3	x_4		
$-z$	1	-3	2	0	0	=	0
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	2

z is not a decision variable

The basic variables are x_3 and x_4.

The non-basic variables are x_1 and x_2.

The basic feasible solution is $x_1 = 0, x_2 = 0, x_3 = 6, x_4 = 2$

Towards: writing an LP in general form.

BV	-z	x_1	x_s	x_3	x_4		
-z	1	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$	=	$-\bar{z}_0$
x_3	0	$\bar{a}_{11}$	$\bar{a}_{12}$	$\bar{a}_{13}$	$\bar{a}_{14}$	=	$\bar{b}_1$
x_4	0	$\bar{a}_{21}$	$\bar{a}_{22}$	$\bar{a}_{23}$	$\bar{a}_{24}$	=	$\bar{b}_2$

The bar indicates that it is the coefficient after some pivots

Use $\bar{c}_j$ to denote the cost coefficients

Use $\bar{b}_i$ to denote the RHS coefficients

Use $\bar{a}_{ij}$ to denote the constraint matrix coefficients.

Towards: writing an LP in general form

BV	-z	x_1	x_2	x_3	x_4		
-z	1	$\bar{c}_1$	$\bar{c}_s$	0	0	=	$-\bar{z}_0$
x_3	0	$\bar{a}_{11}$	$\bar{a}_{1s}$	1	0	=	$\bar{b}_1$
x_4	0	$\bar{a}_{r1}$	$\bar{a}_{rs}$	0	1	=	$\bar{b}_r$

Usually, we write the basic variables as unit vectors

Let s denote the index of the entering variable.

Let r denote the index of the row with the leaving variable.

We pivot on coefficient $\bar{a}_{rs}$

Towards: writing an LP in general form

BV	-z	x_1	x_2	x_3	x_4		
-z	1	$\bar{c}_1$	$\bar{c}_s$	0	0	=	$-\bar{z}_0$
x_3	0	$\bar{a}_{11}$	$\bar{a}_{1s}$	1	0	=	$\bar{b}_1$
x_4	0	$\bar{a}_{r1}$	$\bar{a}_{rs}$	0	1	=	$\bar{b}_r$

To do next: rewrite the LP so that:

(1) the number of equality constraints (rows) is m

(2) the number of variables (columns) is n.

Basic Variables Non-basic Variables

$-z$	x_1	x_2	x_r	x_m	x_{m+1}	x_s	x_n		CV
1	0	0	0	0	$\bar{c}_{m+1}$	$\bar{c}_s$	$\bar{c}_n$	=	$-\bar{z}_0$
0	1	0	0	0	$\bar{a}_{1,m+1}$	$\bar{a}_{1,s}$	$\bar{a}_{1,n}$	=	$\bar{b}_1$
0	0	1	0	0	$\bar{a}_{2,m+1}$	$\bar{a}_{2,s}$	$\bar{a}_{2,n}$	=	$\bar{b}_2$
0	0	0	1	0	$\bar{a}_{r,m+1}$	$\bar{a}_{r,s}$	$\bar{a}_{r,n}$	=	$\bar{b}_r$
0	0	0	0	1	$\bar{a}_{m,m+1}$	$\bar{a}_{m,s}$	$\bar{a}_{m,n}$	=	$\bar{b}_m$

m constraints, n variables

The basic feasible solution

- **The current values are all non-negative.**
  - **This is needed for canonical form**
- **There is a basic variable associated with each constraint.**
  - **in this case, the basic variable associated with constraint i is x_i.**
- **There are n-m nonbasic variables.**
  - **In this case, the nonbasic variables are x_{m+1}, … x_n.**
- **This bfs is as follows: $x_1 = \bar{b}_1$, … , $x_m = \bar{b}_m$. All other variables are 0.**

Optimality Conditions (maximization)

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	-2	-4	0	0	=	-8
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	2

This basic feasible solution is optimal!

What are the optimality conditions, expressed in terms of $\bar{c}$?

Optimality Conditions (maximization)

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	$\bar{c}_1$	$\bar{c}_2$	0	0	=	$-\bar{z}_0$
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	2

This basic feasible solution is optimal!

What are the optimality conditions, expressed in terms of $\bar{c}$?

Pivoting and the min ratio rule

Pivot in variable x_s, where $\bar{c}_s > 0$.

$x_3 = 0$ when $\Delta = 6/3$

$x_4 = 0$ when $\Delta = 2/2$.

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	-3	2	0	0	=	0
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	2

$x_1 = 0$

$x_2 = \Delta$

$x_3 = 6 - 3\Delta$

$x_4 = 2 - 2\Delta$

$z = 2\Delta$

Δ is set to the min(6/3, 2/2) = min $(\bar{b}_1 / \bar{a}_{1s}, \bar{b}_2 / \bar{a}_{2s})$.

Pivoting and the min ratio rule

Pivot in variable x_s, where $\bar{c}_s > 0$.

$x_3 = 0$ when $\Delta = 6/3$

$x_4 = 0$ when $\Delta = 2/2$.

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	0	0	=	0
x_3	0	-3	$\bar{a}_{1s}$	1	0	=	$\bar{b}_1$
x_4	0	-4	$\bar{a}_{2s}$	0	1	=	$\bar{b}_2$

$x_1 = 0$

$x_2 = \Delta$

$x_3 = 6 - 3\Delta$

$x_4 = 2 - 2\Delta$

$z = 2\Delta$

Δ is set to the min(6/3, 2/2) = min $(\bar{b}_1/\bar{a}_{1s}, \bar{b}_2/\bar{a}_{2s})$.

The constraint with a changed basic variable is constraint r, where r = argmin $(\bar{b}_1/\bar{a}_{1s}, \bar{b}_2/\bar{a}_{2s})$ = 2. Min ratio rule.

Express the min ratio rule using general coefficients

Minimum Ratio Rule

Pivot out the basic variable in row r, where

$$r = \text{argmin}_{i} \; \{\bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0\}, \text{ and thus}$$

$$\bar{b}_r / \bar{a}_{rs} = \min \{\bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0\}.$$

If $\bar{a}_{is} \leq 0$ **for all i, then the solution is unbounded.**

Pivoting to obtain a better solution

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	1	0	0	-1	=	-2
x_3	0	3	0	1	-3/2	=	3
x_2	0	-2	1	0	1/2	=	1

$x_1 = 0$
$x_2 = 1$
$x_3 = 3$
$x_4 = 0$
$z = 2$

Pivot in variable x_s, where $\bar{c}_s > 0$.

Pivot out the basic variable for constraint r according to the min ratio rule.

Alternative Optima (maximization)

Recall: $z = 8 + 0\ x_1 + x_2$

How much can x_1 be increased?

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	0	-4	0	0	=	-8
x_3	0	1	3	1	0	=	6
x_4	0	-1	2	0	1	=	2

This basic feasible solution is optimal! Is there another optimal solution?

Suppose that x_1 entered the basis.
What would leave?

Alternative Optima (maximization)

BV	-z	x_1	x_2	x_3	x_4		
-z	1	$\bar{c}_1$	$\bar{c}_2$	0	0	=	-8
x_3	0	1	3	1	0	=	6
x_4	0	-1	2	0	1	=	2

There may be alternative optima if $\bar{c}_j \leq 0$ for all j and $\bar{c}_j = 0$ for some j where x_j is non-basic

Use the min ratio rule to determine which variable leaves the basis.

Alternative Optima (maximization)

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	0	-4	0	0	=	-8
x_1	0	1	3	1	0	=	6
x_4	0	0	5	1	1	=	8

Let x_1 enter the basis. Let the basic variable in constraint 1 leave the basis.

Note: the solution is different, but the objective value is the same.

Unboundedness

Express unboundedness in terms of algebraic notation.

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	1	0	0	-1	=	-2
x_3	0	-3	0	1	-1.5	=	3
x_2	0	-2	1	0	.5	=	1

$x_1 = \Delta$

$x_2 = 1 + 2\Delta$

$x_3 = 3 + 3\Delta.$

$x_4 = 0$

$z = 2 + \Delta$

The entering variable is x_1.

Set $x_1 = \Delta$ and $x_4 = 0$.

If the coefficients in the entering column are ≤ 0, then the solution is ***unbounded*** from above

Review of notation

A basic feasible solution is optimal if $\bar{c}_j \leq 0$ for all j.

Assumption: the entering variable is x_s (and so $\bar{c}_s > 0$)

Pivot out the basic variable in row r, where

$$r = \text{argmin}_i \ \{\bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0\}, \text{ and thus}$$

$$\bar{b}_r / \bar{a}_{rs} = \min \{\bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0\}.$$

If $\bar{a}_{is} \leq 0$ for all i, then the solution is unbounded.

Simplex Method (Max Form)

Step 0. The problem is in canonical form and $\bar{b} \geq 0$.

Step 1. If $\bar{c} \leq 0$ then stop. The solution is optimal. If we continue, then there exists some $\bar{c}_j > 0$.

Step 2. Choose any non-basic variable to pivot in with $\bar{c}_s > 0$, e.g., $\bar{c}_s = \max_j \{\bar{c}_j \mid \bar{c}_j > 0\}$. If $\bar{a}_{is} \leq 0$ for all i, then stop; the LP is unbounded. If we continue, then there exists some $\bar{a}_{is} > 0$.

Step 3. Pivot out the basic variable in row r, where r is chosen by the min ratio rule, that is $r = \operatorname{argmin}_i (\bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0)$.

Step 4. Replace the basic variable in row r with variable x_s and re-establish canonical form (i.e., pivot on the coefficient $\bar{a}_{rs}$.)

Step 5. Go to Step 1.

Preview of what is to come

- Degeneracy and improving solutions
- Proving finiteness and optimality under no degeneracy
- Handling degeneracy
- Obtaining an initial canonical form

Degeneracy

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	0	0	=	-2
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	0

A bfs is ***degenerate*** if $\bar{b}_j = 0$ for some j.

Otherwise, it is ***non-degenerate***.

This bfs is *degenerate*.

Degeneracy

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	0	0	=	-2
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	3

A bfs is *degenerate* if $\bar{b}_j = 0$ for some j.

Otherwise, it is *non-degenerate*.

This bfs is *non-degenerate*.

Degeneracy

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	-3	2	0	0	=	-2
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	0

$x_1 = 0$

$x_2 = \Delta$

$x_3 = 6 - 3\Delta$

$x_4 = 0 - 2\Delta$

$z = 2 + 2\Delta$

A bfs is *degenerate* if $\bar{b}_j = 0$ for some j.

Otherwise, it is *non-degenerate*.

Suppose that variable x_2 will enter the basis.

Degenerate ⇒ the solution may stay the same

⇒ z stays the same

A degenerate pivot

BV	-z	x_1	x_2	x_3	x_4		
-z	1	1	0	0	-1	=	-2
x_3	0	3	0	1	-3/2	=	6
x_4	0	-2	1	0	1/2	=	0

$x_1 = 0$
$x_2 = 0$
$x_3 = 6$
$x_4 = 0$
$z = 2$

The entering variable is x_2. The exiting variable is the one in constraint 2.

In this case the solution did not change.
(But the tableau did.)

Error on this slide. Where?

A degenerate pivot

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	1	0	0	-1	=	-2
x_3	0	3	0	1	-3/2	=	6
x_2	0	-2	1	0	1/2	=	0

$x_1 = 0$
$x_2 = 0$
$x_3 = 6$
$x_4 = 0$
$z = 2$

The entering variable is x_2. The exiting variable is the one in constraint 2.

In this case the solution did not change. (But the tableau did.)

Non-degeneracy leads to strict improvement

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	0	0	=	-2
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	2

$x_1 = 0$

$x_2 = \Delta$

$x_3 = 6 - 3\Delta$

$x_4 = 2 - 2\Delta$

$z = 2 + 2\Delta$

If the bfs is non-degenerate, then the entering variable can strictly increase, and the objective value *will* strictly improve.

Theorem. If every basic feasible solution is non-degenerate, the simplex method is finite.

1. The number of basic feasible solutions is at most n! / (n-m)! m! , which is the number of ways of selecting m basic variables out of n.

1. Each basic feasible solution is different
   - Each has a strictly better cost than the last, assuming non-degeneracy

- Therefore, the simplex method is finite

Handling Degeneracy

- Perturb the RHS just a little, in just the right way
  - No basis is degenerate
  - Every bfs to the purturbed problem is also a bfs for the original problem
  - The optimal basis for the perturbed problem is optimal for the original problem

An LP before perturbation

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	-3	2	0	0	=	0
x_3	0	-3	3	1	0	=	6.
x_4	0	-4	2	0	1	=	2.

Goal: modify the problem by changing the RHS just a little

Perturb it to avoid degeneracy

Perturb it so little that solving the perturbed problem also solves the original problem.

An example of a perturbed initial bfs

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	-3	2	0	0	=	0
x_3	0	-3	3	1	0	=	6.00000000000013
x_4	0	-4	2	0	1	=	2.00000000000041

In theory perturbations are even smaller

In practice, the simplex method works fine without perturbations, that is, it really obtains the optimum basic feasible solution.

Some Remarks on Degeneracy and Alternative Optima

- **It might seem that degeneracy would be rare. After all, why should we expect that the RHS would be 0 for some variable?**
- **In reality, degeneracy is incredibly common.**
- **The simplex method is not necessarily finite unless care is taken in the pivot rule. In reality, almost no one takes care, and the simplex method is not only finite, but it is incredibly efficient.**
- **The issue of alternative optima arises frequently, and is of importance in practice.**

Stop here for now

Simplex Algorithm: getting started

- To start the simplex algorithm, we need a canonical form, which corresponds to a basic feasible solution. So, how do we get started?

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	4	1	=	0
?	0	-3	3	2	5	=	6
?	0	-4	2	1	3	=	2

The origin is not feasible: $x_1 = x_2 = x_3 = x_4 = 0$ violates the constraints.

A naïve suggestion: just choose the variables and then pivot to get into canonical form.

This technique sometimes works, but ….

Suppose we try to make x_3 and x_4 the basic variables

The tableau after bringing it to Jordan-canonical form:

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-42	5	0	0	=	-30
x_3	0	11	-1	1	0	=	8
x_4	0	-5	1	0	1	=	-2

So, guessing may not create a basic feasible solution.

There may not even be a feasible solution!

A suggestion that sounds dumb, but almost works: let's create an artificial basis.

Let's just make up some variables and use them to get started.

BV	-z	x_1	x_2	x_3	x_4	x_5	x_6		
-z	1	-3	2	4	1	0	0	=	0
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Obvious difficulty: we are now solving a different problem. There is no guarantee that this will help us solve our original problem.

Trying to make artificial bases work

- **New model: original model plus artificial variables. The artificial variables help us get started.**

- **What we want: optimizing the new model will produce an optimum solution to the original model.**

- **Potential Danger: the artificial variables will be positive when optimizing the new model. This would be bad.**

- **So, we can add artificial variables to our model, if we can guarantee that the variables take on a value of 0 in the optimum solution.**

So, how can we modify x_5 and x_6 so that they won't be in an optimal solution?

Give them a large cost. If the cost is big enough, then x_5 and x_6 will not be positive in an optimum solution

BV	-z	x_1	x_2	x_3	x_4	x_5	x_6		
-z	1	-3	2	4	1	-50	-50	=	0
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Two issues: (1) the problem is no longer in canonical form

(2) will an optimum for this model also be optimum for the original?

Getting into canonical form

Add 50 times constraint 1 and 50 times constraint 2 to the cost coefficients

BV	$-z$	x_1	x_2	x_3	x_4	x_5	x_6		
$-z$	1	-353	252	154	401	0	0	=	400
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Remaining issue: Will an optimum for this model also be optimum for the original?

3 pivots later

We obtain an optimal basic feasible solution to the new problem.

BV	-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
-z	1	0	-3.4	0	-8.4	-52.6	-48.8	=	-13.2
x_1	0	1	-.2	0	-.2	.2	-.4	=	.4
x_3	0	0	1.2	1	2.2	.8	.6	=	3.6

Eliminating x_5 and x_6 creates an optimal solution to the original problem (indeed $x_5 = x_6 = 0$ in the solution).

3 pivots later

We obtain an optimal basic feasible solution to the new problem.

BV	-z	x_1	x_2	x_3	x_4		
-z	1	0	-3.4	0	-8.4	=	-13.2
x_1	0	1	-.2	0	-.2	=	.4
x_3	0	0	1.2	1	2.2	=	3.6

Eliminating x_5 and x_6 creates an optimal solution to the original problem (indeed $x_5 = x_6 = 0$ in the solution).

The Big M method

- The cost coefficient of the artificial variables should be –M for some large value of M. (We used M = 50)
- We want to guarantee that none of these variables should be positive in an optimum solution

- Difficulties.
  - How big should M be?
  - Large values of M can increase the problem of numerical round-off errors (also known as numerical instability)

An alternative approach: The Phase 1 method

- Recall: we used artificial variables to get the simplex started.
- Any basic feasible solution would get the simplex method started
- We can add artificial variables, and then focus entirely on obtaining a basic feasible solution, any basic feasible solution.
- We can then start the simplex algorithm with the basic feasible solution we have found

Observation 1: if all we want is a basic feasible solution, then we can select any objective function.

-z	x_1	x_2	x_3	x_4		
1	?	?	?	?	=	?
0	-3	3	2	5	=	6
0	-4	2	1	3	=	2

FACT: Once we find a basic feasible solution, we can reconsider the original cost coefficients.

It's easy to bring cost coefficients into canonical form.

We will choose an objective function soon.

Next: add in the artificial variables, creating a basic feasible solution to the new problem

-z	x_1	x_2	x_3	x_4	x_5	x_6		
1	?	?	?	?	?	?	=	?
0	-3	3	2	5	1	0	=	6
0	-4	2	1	3	0	1	=	2

Now: choose an objective such that x_5 and x_6 are guaranteed to be 0 if we optimize the objective.

Minimize $x_5 + x_6$. Or maximize $w = -x_5 - x_6$.

The phase 1 problem: almost in canonical form

-z	x_1	x_2	x_3	x_4	x_5	x_6		
1	0	0	0	0	-1	-1	=	0
0	-3	3	2	5	1	0	=	6
0	-4	2	1	3	0	1	=	2

To get into canonical form, add constraints 1 and 2 to the objective.

The phase 1 problem: now in canonical form

BV	-z	x_1	x_2	x_3	x_4	x_5	x_6		
-z	1	-7	5	3	8	0	0	=	8
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

We are now in a form to start the simplex algorithm

Two pivots later, we have an optimal solution to the phase 1 problem.

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6		
-w	1	0	0	0	0	-1	-1	=	0
x_1	0	1	0	1/6	1/6	1/3	-1/2	=	1
x_2	0	0	1	5/6	11/6	2/3	-1/2	=	3

So, we have identified a basic feasible solution with basic variables x_1 and x_2.

This leads to a bfs for our original problem

We now need to return to our original problem, and continue to find the optimum

Recovering a bfs for the original problem

BV	-w	x_1	x_2	x_3	x_4		
-w	1	0	0	0	0	=	0
x_1	0	1	0	1/6	1/6	=	1
x_2	0	0	1	5/6	11/6	=	3

At the end of phase 1, eliminate the "artificial variables" x_5 and x_6.

Recovering a bfs for the original problem

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	4	1	=	0
x_1	0	1	0	1/6	1/6	=	1
x_2	0	0	1	5/6	11/6	=	3

At the end of phase 1, eliminate the "artificial variables" x_5 and x_6.

Reintroduce the original objective

Recovering a bfs for the original problem

BV	-z	x_1	x_2	x_3	x_4		
-z	1	0	0	17/6	-13/6	=	-3
x_5	0	1	0	1/6	1/6	=	1
x_6	0	0	1	5/6	11/6	=	3

At the end of phase 1, eliminate the "artificial variables" x_5 and x_6.
Reintroduce the original objective

Then bring into canonical form. We can now start *phase 2.*

Phase 2. And one pivot later

$-z$	x_1	x_2	x_3	x_4		
1	0	-3.4	0	-8.4	=	-13.2
0	1	-.2	0	-.2	=	.4
0	0	1.2	1	2.2	=	3.6

We have now solved the original problem.

Phase 1 seems like a lot of work. It can do a lot of pivots, and the only purpose is to find a basic feasible solution, so that we can start "phase 2"

It's what is done in practice

It works very well.

Summary

- To get started with the simplex method, add an artificial basis, but ensure that these artificial variables don’t occur in an optimal solution.
- Big M method: put a large cost on each of the artificial variables
- Phase 1 method. Minimize the sum of the artificial variables. At the end of phase 1, we will have a basic feasible solution to the original problem. Use this as a starting point for Phase 2, which solves the original problem.

The following are extra slides

Creating a Phase 1 Problem

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	4	1	=	0
x_5	0	-3	3	2	5	=	6
x_6	0	-4	2	1	3	=	2

Eliminate the objective function, for now

Creating a Phase 1 Problem

BV	-z	x_1	x_2	x_3	x_4		
x_5	0	-3	3	2	5	=	6
x_6	0	-4	2	1	3	=	2

Eliminate the objective function, for now

Add the artificial variables

Creating a Phase 1 Problem

BV	-z	x_1	x_2	x_3	x_4	x_5	x_6		
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Eliminate the objective function, for now

Add the artificial variables

Minimize the sum of the artificials

Creating a Phase 1 Problem

BV	$-w$	x_1	x_2	x_3	x_4	x_5	x_6		
$-w$	1	0	0	0	0	-1	-1	=	0
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Eliminate the objective function, for now

Add the artificial variables

Minimize the sum of the artificials

Bring into canonical form

Creating a Phase 1 Problem

BV	$-w$	x_1	x_2	x_3	x_4	x_5	x_6		
$-w$	1	-7	5	3	8	0	0	=	8
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Eliminate the objective function, for now

Add the artificial variables

Minimize the sum of the artificials

Bring into canonical form

Creating a bfs from the phase 1 solution

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6		
-w	1	0	0	0	0	-1	-1	=	0
x_1	0	1	0	1/6	1/6	1/3	-1/2	=	1
x_2	0	0	1	5/6	11/6	2/3	-1/2	=	3

At end, if w > 0, report that there is no feasible solution.

If w = 0, eliminate artificial variables (or keep the columns but forbid them from pivoting in).

Creating a bfs from the phase 1 solution

BV	-w	x_1	x_2	x_3	x_4		
-w	1	0	0	0	0	=	0
x_1	0	1	0	1/6	1/6	=	1
x_2	0	0	1	5/6	11/6	=	3

At end, if w > 0, report that there is no feasible solution.

If w = 0, eliminate artificial variables (or keep the columns but forbid them from pivoting in).

Reintroduce the original objective

Creating a bfs from the phase 1 solution

BV	-z	x_1	x_2	x_3	x_4		RHS
-z	1	-3	2	4	1	=	0
x_1	0	1	0	1/6	1/6	=	1
x_2	0	0	1	5/6	11/6	=	3

At end, if w > 0, report that there is no feasible solution.

If w = 0, eliminate artificial variables (or keep the columns but forbid them from pivoting in).

Reintroduce the original objective

Then bring into canonical form.

Creating a bfs from the phase 1 solution

BV	-z	x_1	x_2	x_3	x_4		
-z	1	0	0	17/6	-13/6	=	-3
x_1	0	1	0	1/6	1/6	=	1
x_2	0	0	1	5/6	11/6	=	3

At end, if w > 0, report that there is no feasible solution.

If w = 0, eliminate artificial variables (or keep the columns but forbid them from pivoting in).

Reintroduce the original objective

Then bring into canonical form. This begins Phase 2.

Potential Difficulties

- If the original problem is degenerate, it is possible that an artificial is in the basis at the end of phase 1.
  - Solution: pivot out the artificial variable
- If the original problem has a redundant constraint, then it is possible that an artificial variable is in the basis at the end of phase 1, and cannot be pivoted out.
  - Solution: eliminate (or ignore) the redundant constraint

Summary

- Review of the simplex algorithm.
- Degeneracy and Alternative Optimal Solutions
- Is the simplex algorithm finite? (Answer, yes, but only if we are careful)
- How do we get an initial BFS?
  - Phase I Approach

Phase 1: obtaining an initial bfs

- We know how to obtain an optimal bfs if we are given an initial bfs.

- To create an initial bfs for problem P, we will use the simplex algorithm on problem P', which is closely connected to P.

Phase 1: obtaining an initial bfs

- **We know how to obtain an optimal bfs if we are given an initial bfs.**

- **To create an initial bfs for problem P, we will use the simplex algorithm on problem P', which is closely connected to P.**

How does one obtain an initial bfs?

-z	x_1	x_2	x_3	x_4		
1	-3	2	4	1	=	0
0	-3	3	2	5	=	6
0	-4	2	1	3	=	2

FACT: Obtaining an initial basic feasible solution is (theoretically) as difficult as obtaining an optimal one

IDEA: Use the simplex algorithm to obtain an initial bfs.

Reducing to a Previously Solved Problem

Juan's Problem

Find a feasible solution to

$$-3\,x_1 + 3\,x_2 + 2\,x_3 + 5\,x_4 = 6$$

$$-4\,x_1 + 2\,x_2 + 1\,x_3 + 3\,x_4 = 2$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4$$

Maria's Problem

Minimize $x_5 + x_6$

subject to:

$$-3\,x_1 + 3\,x_2 + 2\,x_3 + 5\,x_4 + x_5 = 6$$

$$-4\,x_1 + 2\,x_2 + 1\,x_3 + 3\,x_4 + x_6 = 2$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5, 6$$

Maria's Problem

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6		
-w	1	0	0	0	0	-1	-1	=	0
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Maria's Problem

Here is a feasible solution to Maria's Problem.

BV	-w	1	3	0	0	0	0		
-w	1	0	0	0	0	-1	-1	=	0
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Recall: we want to minimize $w = x_5 + x_6$

No solution has a cost less than 0 (x_5 and $x_6 \geq 0$).

Juan's Problem

To get an feasible solution to Juan's problem, drop x_5 and x_6.

BV	-w						
BV	-w	1	3	0	0		
-w	1	0	0	0	0	=	0
x_5	0	-3	3	2	5	=	6
x_6	0	-4	2	1	3	=	2

Conclusion: Juan's Problem and Maria's Problem are equivalent.

Not so obvious: Solving Maria's Problem using simplex will yield a bfs for Juan.

Maria's Problem in Tableau Form

BV	$-w$	x_1	x_2	x_3	x_4	x_5	x_6		
$-w$	1	0	0	0	0	-1	-1	=	0
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

This problem is not yet in canonical form. But it is close.

What transformation do we need to do?

Maria's Problem in Canonical Form

Add constraints 1 and 2 to the objective.

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6		
-w	1	-7	5	3	8	0	0	=	8
x_5	0	-3	3	2	5	1	0	=	6
x_6	0	-4	2	1	3	0	1	=	2

Then run the simplex algorithm on Maria's problem

It will find a feasible solution (if one exists) for Juan's problem, and it will terminate with a bfs.

The optimal basis for Maria's Problem

The optimal basic variables are x_1 and x_2.

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6		
-w	1	0	0	0	0	-1	-1	=	0
x_5	0	1	0	1/6	1/6	1/3	-1/2	=	1
x_6	0	0	1	5/6	11/6	2/3	-1/2	=	3

At the end, one can obtain a bfs for the original problem by dropping x_5 and x_6.