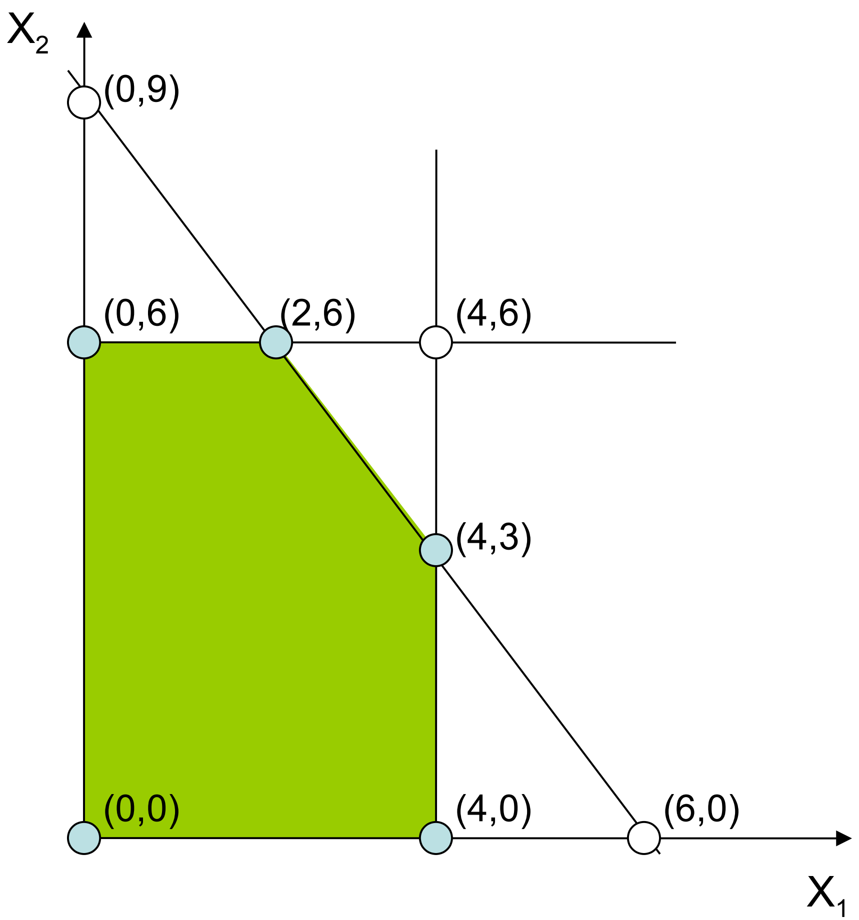
Solving Linear Programs: The Simplex Method

The Essence

- Simplex method is an algebraic procedure
- However, its underlying concepts are geometric
- Understanding these geometric concepts helps before going into their algebraic equivalents

Back to Wyndor Glass



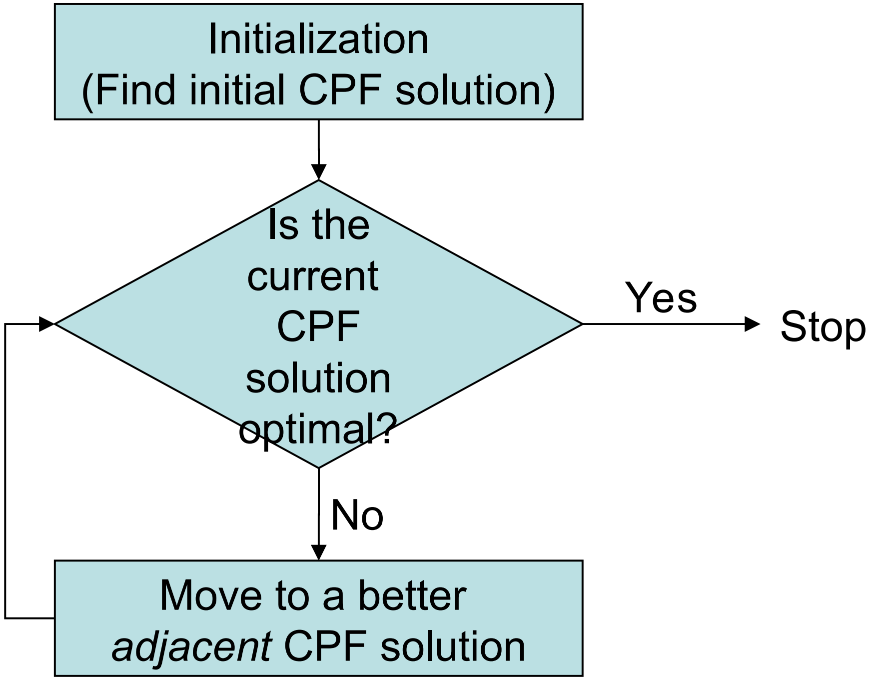
- Constraint boundaries
- Feasible region
- Corner-point solutions
- Corner-point feasible (CPF) solutions
- Adjacent CPF solutions
- Edges of the feasible region

Optimality test in the Simplex Method:

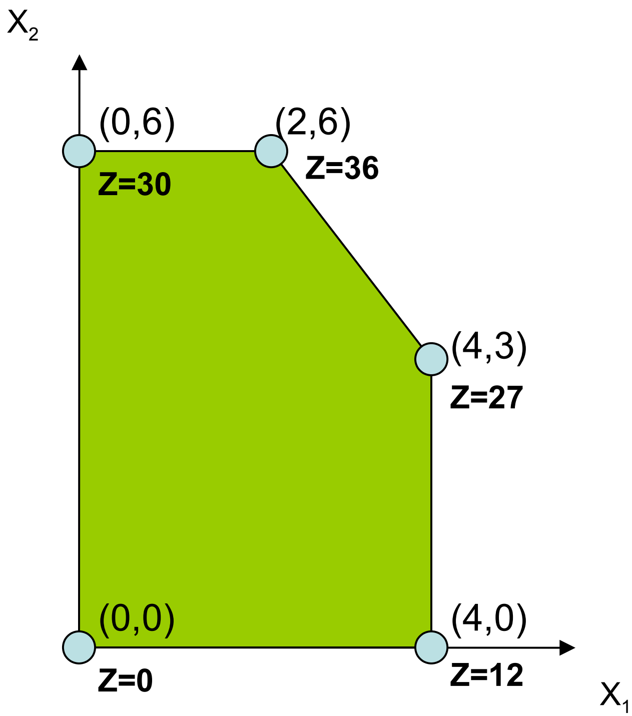
If a CPF solution has no adjacent solutions that are better, then it *must* be an optimal solution

The Simplex Method in a Nutshell

An *iterative* procedure



Solving Wyndor Glass



- Initial CPF
- Optimality test
- If not optimal, then move to a better adjacent CPF solution:
  - Consider the edges that emanate from current CPF
  - Move along the edge that increases Z at a faster rate
  - Stop at the first constraint boundary
  - Solve for the intersection of the new boundaries
  - Back to optimality test

Key Concepts

- Focus only on CPF solutions
- An iterative algorithm
- If possible, use the origin as the initial CPF solution
- Move always to adjacent CPF solutions
- Don't calculate the Z value at adjacent solutions, instead move directly to the one that 'looks' better (on the edge with the higher rate of improvement)
- Optimality test also looks at the rate of improvement (If all negative, then optimal)

‘Language’ of the Simplex Method

Initial Assumptions

- All constraints are of the form ≤
- All right-hand-side values (b_j, $j=1, \ldots, m$) are positive
- We'll learn how to address other forms later

The Augmented Form

Set up the method first:
Convert inequality constraints to equality constraints by adding slack variables

Original Form

Maximize $Z = 3x_1 + 5x_2$

subject to

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

***Augmented** Form*

Maximize $Z = 3x_1 + 5x_2$

subject to

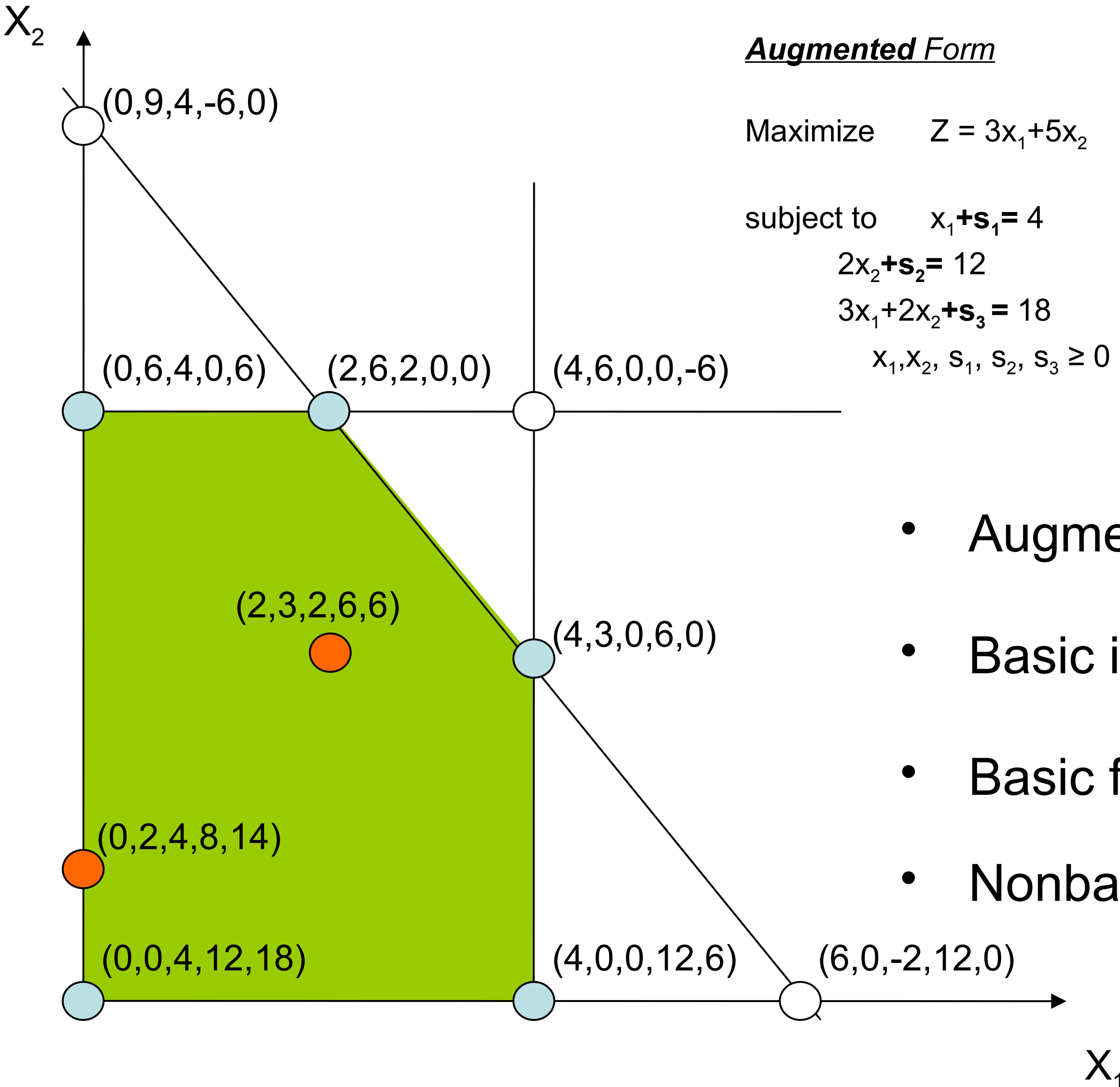
$x_1 + \mathbf{s_1} = 4$

$2x_2 + \mathbf{s_2} = 12$

$3x_1 + 2x_2 + \mathbf{s3} = 18$

$x_1, x_2 \geq 0$

Basic and Basic Feasible Solutions



***Augmented** Form*

Maximize $Z = 3x_1+5x_2$

subject to $x_1 + \mathbf{s_1} = 4$

$2x_2 + \mathbf{s_2} = 12$

$3x_1+2x_2 + \mathbf{s_3} = 18$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

- Augmented solution
- Basic infeasible solution
- Basic feasible solution (BFS)
- Nonbasic feasible solution

Basic, Nonbasic Solutions and the Basis

- In an LP, number of variables > number of equations
- The difference is the degrees of freedom of the system
  - e.g. in Wyndor Glass, degrees of freedom (d.f.)=
- Can set some variables (# = d.f.) to an arbitrary value (simplex uses 0)
- These variables (set to 0) are called *nonbasic* variables
- The rest can be found by solving the remaining system
- *The basis*: the set of basic variables
- If all basic variables are ≥ 0, we have a BFS
- Between two basic solutions, if their bases are the same except for one variable, then they are adjacent

Basis Examples: Wyndor Glass

Maximize	$Z =$	$3x_1 +$	$5x_2$				
subject to		x_1		$+s_1$			$= 4$
			$2x_2$		$+s_2$		$= 12$
		$3x_1 +$	$2x_2$			$+s_3$	$= 18$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

- If the basis was:

$(x_1, x_2, s_1) \rightarrow$

$(x_1, x_2, s_2) \rightarrow$

$(s_1, s_2, s_3) \rightarrow$

- Which ones are BFS? Which pairs are adjacent?

Algebra of the Simplex Method

Initialization

Maximize $Z = 3x_1 + 5x_2$

subject to

$$
\begin{aligned}
x_1 \quad\quad\quad + \mathbf{s_1} \quad\quad\quad &= 4 \\
2x_2 \quad\quad + \mathbf{s_2} \quad &= 12 \\
3x_1 + 2x_2 \quad\quad\quad + \mathbf{s_3} &= 18
\end{aligned}
$$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

- Find an initial basic feasible solution
- Remember from key concepts:
  "If possible, use the origin as the initial CPF solution"
- Equivalent to:
  Choose original variables to be nonbasic (x_i=0, i=1,…n) and let the slack variables be basic (s_j=b_j, j=1,…m))

Algebra of the Simplex Method

Optimality Test

Maximize $Z = 3x_1 + 5x_2$

subject to

$$x_1 + \mathbf{s_1} = 4$$
$$2x_2 + \mathbf{s_2} = 12$$
$$3x_1 + 2x_2 + \mathbf{s_3} = 18$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

- Are any adjacent BF solutions better than the current one?
- Rewrite Z in terms of <u>nonbasic</u> variables and investigate rate of improvement
- Current nonbasic variables:
- Corresponding Z:
- Optimal?

Algebra of the Simplex Method

Step 1 of Iteration 1: Direction of Movement

$$
\begin{array}{llllllll}
\text{Maximize} & Z = & 3x_1 + & 5x_2 & & & & \\
\text{subject to} & & x_1 & & +\mathbf{s_1} & & & = 4 \\
 & & & 2x_2 & & +\mathbf{s_2} & & = 12 \\
 & & 3x_1 + & 2x_2 & & & +\mathbf{s_3} & = 18 \\
 & & \multicolumn{6}{l}{x_1, x_2, s_1, s_2, s_3 \geq 0}
\end{array}
$$

- Which edge to move on?
- Determine the direction of movement by selecting the ***entering variable*** (variable ‘entering’ the basis)
- Choose the direction of steepest ascent
  - x_1: Rate of improvement in Z =
  - x_2: Rate of improvement in Z =
- Entering basic variable =

Algebra of the Simplex Method

Step 2 of Iteration 1: Where to Stop

Maximize $Z = 3x_1 + 5x_2$

subject to

$x_1 \quad\quad + \mathbf{s_1}$	$= 4$	(1)
$2x_2 \quad + \mathbf{s_2}$	$= 12$	(2)
$3x_1 + 2x_2 \quad + \mathbf{s_3}$	$= 18$	(3)

$x_1, x_2, s_1, s_2, s_3 \geq 0$

- How far can we go?
- Determine where to stop by selecting the ***leaving variable*** (variable 'leaving' the basis)
- Increasing the value of x_2 decreases the value of basic variables
- The **minimum ratio test**
  - Constraint (1):
  - Constraint (2):
  - Constraint (3):
- Leaving basic variable =

Algebra of the Simplex Method

Step 3 of Iteration 1: Solving for the New BF Solution

$Z -$	$3x_1 -$	$5x_2$				$= 0$	(0)
	x_1		$+s_1$			$= 4$	(1)
		$2x_2$		$+s_2$		$= 12$	(2)
	$3x_1 +$	$2x_2$			$+s_3$	$= 18$	(3)

- Convert the system of equations to a more proper form for the new BF solution
- Elementary algebraic operations: *Gaussian elimination*
  - **Eliminate the entering basic variable (x_2) from all but *its* equation**

Algebra of the Simplex Method

Optimality Test

$Z -$	$3x_1 +$			$+ 5/2\, s_2$		$= 30$	(0)
	x_1		$+s_1$			$= 4$	(1)
		x_2		$+ 1/2\, s_2$		$= 6$	(2)
	$3x_1$			$- s_2$	$+ s_3$	$= 6$	(3)

- Are any adjacent BF solutions better than the current one?
- Rewrite Z in terms of <u>nonbasic</u> variables and investigate rate of improvement
- Current nonbasic variables:
- Corresponding Z:
- Optimal?

Algebra of the Simplex Method

Step 1 of Iteration 2: Direction of Movement

$$
\begin{array}{llllllll}
Z - & 3x_1 + & & & + 5/2\, s_2 & & = 30 & (0) \\
 & x_1 & & + s_1 & & & = 4 & (1) \\
 & & x_2 & & + 1/2\, s_2 & & = 6 & (2) \\
 & 3x_1 & & & - s_2 & + s_3 & = 6 & (3)
\end{array}
$$

- Which edge to move on?
- Determine the direction of movement by selecting the ***entering variable*** (variable 'entering' the basis)
- Choose the direction of steepest ascent
  - x_1: Rate of improvement in Z =
  - s_2: Rate of improvement in Z =
- Entering basic variable =

Algebra of the Simplex Method

Step 2 of Iteration 2: Where to Stop

$Z -$	$3x_1 +$			$+ 5/2\ s_2$		$= 30$	(0)
	x_1		$+s_1$			$= 4$	(1)
		x_2		$+ 1/2\ s_2$		$= 6$	(2)
	$3x_1$			$- s_2$	$+ s_3$	$= 6$	(3)

- How far can we go?
- Determine where to stop by selecting the ***leaving variable*** (variable 'leaving' the basis)
- Increasing the value of x_1 decreases the value of basic variables
- The **minimum ratio test**
  - Constraint (1):
  - Constraint (2):
  - Constraint (3):
- Leaving basic variable =

Algebra of the Simplex Method

Step 3 of Iteration 2: Solving for the New BF Solution

$$
\begin{array}{llllllll}
Z - & 3x_1 + & & & + 5/2\, s_2 & & = 30 & (0) \\
 & & & & & & & \\
 & x_1 & & + s_1 & & & = 4 & (1) \\
 & & x_2 & & + 1/2\, s_2 & & = 6 & (2) \\
 & 3x_1 & & & - s_2 & + s_3 & = 6 & (3)
\end{array}
$$

- Convert the system of equations to a more proper form for the new BF solution
- Elementary algebraic operations: *Gaussian elimination*
  - **Eliminate the entering basic variable (x_1) from all but *its* equation**

Algebra of the Simplex Method

Optimality Test

Z				$+ 3/2\, s_2$	$+ s_3$	$= 36$	(0)
			$+s_1$	$+ 1/3\, s_2$	$- 1/3\, s_3$	$= 2$	(1)
		x_2		$+ 1/2\, s_2$		$= 6$	(2)
	x_1			$- 1/3\, s_2$	$+ 1/3\, s_3$	$= 2$	(3)

- Are any adjacent BF solutions better than the current one?
- Rewrite Z in terms of <u>nonbasic</u> variables and investigate rate of improvement
- Current nonbasic variables:
- Corresponding Z:
- Optimal?

The Simplex Method in Tabular Form

- For convenience in performing the required calculations
- Record only the essential information of the (evolving) system of equations in tableaux
  - Coefficients of the variables
  - Constants on the right-hand-sides
  - Basic variables corresponding to equations

Wyndor Glass

$$
\begin{array}{lllllll}
Z - & 3x_1 - & 5x_2 & & & & = 0 & (0) \\
 & x_1 & & +s_1 & & & = 4 & (1) \\
 & & 2x_2 & & +s_2 & & = 12 & (2) \\
 & 3x_1 + & 2x_2 & & & +s_3 & = 18 & (3)
\end{array}
$$

- Convert to initial tableau

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.

Wyndor Glass, Iteration 1

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	-3	-5	0	0	0	0
s_1	0	1	0	1	0	0	4
s_2	0	0	2	0	1	0	12
s_3	0	3	2	0	0	1	18

- Optimality test
- Entering variable (steepest ascent) – pivot column
- Leaving variable (minimum ratio test) – pivot row
- Gaussian elimination

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1						
	0						
	0						
	0						

Wyndor Glass, Iteration 2

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	-3	0	0	5/2	0	30
s_1	0	1	0	1	0	0	4
x_2	0	0	1	0	1/2	0	6
s_3	0	3	0	0	-1	1	6

- Optimality test
- Entering variable (steepest ascent) – pivot column
- Leaving variable (minimum ratio test) – pivot row
- Gaussian elimination

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1						
	0						
	0						
	0						

Wyndor Glass, Iteration 3

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	0	0	0	3/2	1	36
s_1	0	0	0	1	1/3	-1/3	2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2

- Optimality test
- Entering variable (steepest ascent) – pivot column
- Leaving variable (minimum ratio test) – pivot row
- Gaussian elimination

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1						
	0						
	0						
	0						

Special Cases, Example 1

$$Z - 3x_1 - 5x_2 = 0 \quad (0)$$

$$x_1 + s_1 = 4 \quad (1)$$

Basic variable	Z	x_1	x_2	s_1	r.h.s.

Basic variable	Z	x_1	x_2	s_1	r.h.s.

Special Cases, Example 2

$Z -$	$6x_1 -$	$4x_2$				$= 0$	(0)

$$Z - 6x_1 - 4x_2 = 0 \quad (0)$$

$$x_1 + s_1 = 4 \quad (1)$$

$$2x_2 + s_2 = 12 \quad (2)$$

$$3x_1 + 2x_2 + s_3 = 18 \quad (3)$$

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.

Special Cases, Example 2, cont'd

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.

Special Cases, Example 3

$$Z - 3x_1 - 3x_2 = 0 \quad (0)$$

$$x_1 + s_1 = 4 \quad (1)$$

$$2x_2 + s_2 = 12 \quad (2)$$

$$3x_1 + 2x_2 + s_3 = 18 \quad (3)$$

Basic variable	**Z**	**x_1**	**x_2**	**s_1**	**s_2**	**s_3**	**r.h.s.**

Special Cases, Example 4

$$Z - 3x_1 - 5x_2 = 0 \quad (0)$$

$$x_1 + s_1 = 4 \quad (1)$$

$$2x_2 + s_2 = 12 \quad (2)$$

$$3x_1 + 3x_2 + s_3 = 18 \quad (3)$$

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.

Special Cases, Summary

- If no variable qualifies to be the leaving variable, then the LP is *unbounded*
- If the Z-row coefficient of a nonbasic variable is zero, the variable may enter the basis, however the objective function will not change
  - If, in addition, coefficients of all other nonbasic variables are ≥ 0, then there are *multiple optimal solutions*
- If there is a tie for the entering variable, break it arbitrarily
  - It will only affect the path taken, but the same optimal solution will be reached
- If there is a tie for the leaving variable, theoretically the way in which the tie is broken is important
  - The method can get trapped in an infinite loop *(cycling under degeneracy)*
  - Beyond the scope of this class

Other Problem Forms

- Until now, we assumed
  - Only constraints of the form $\leq$
  - Only positive right-hand-side values ($b_j \geq 0$)
  - Non-negativity
- We relaxed the maximization assumption earlier:

  min cx =
- Time to relax the other assumptions

Equality Constraints

- Consider the Wyndor Glass problem, where Plant 3 constraint is changed as follows:

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 5x_2 \\ \text{subject to} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 = 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Convert to augmented form:

- Any problems you foresee with the simplex method?

Artificial Variables and the Big M

- Introduce artificial variables to the problem
- Assign huge penalties in the objective function

$$\begin{array}{llll} \text{Maximize} & Z = & 3x_1 + 5x_2 & \\ \text{subject to} & & x_1 & \leq 4 \\ & & 2x_2 & \leq 12 \\ & & 3x_1 + 2x_2 & = 18 \\ & & x_1, x_2 \geq 0 & \end{array}$$

Solving the new problem (1)

Basic variable	Z	x_1	x_2	r.h.s.

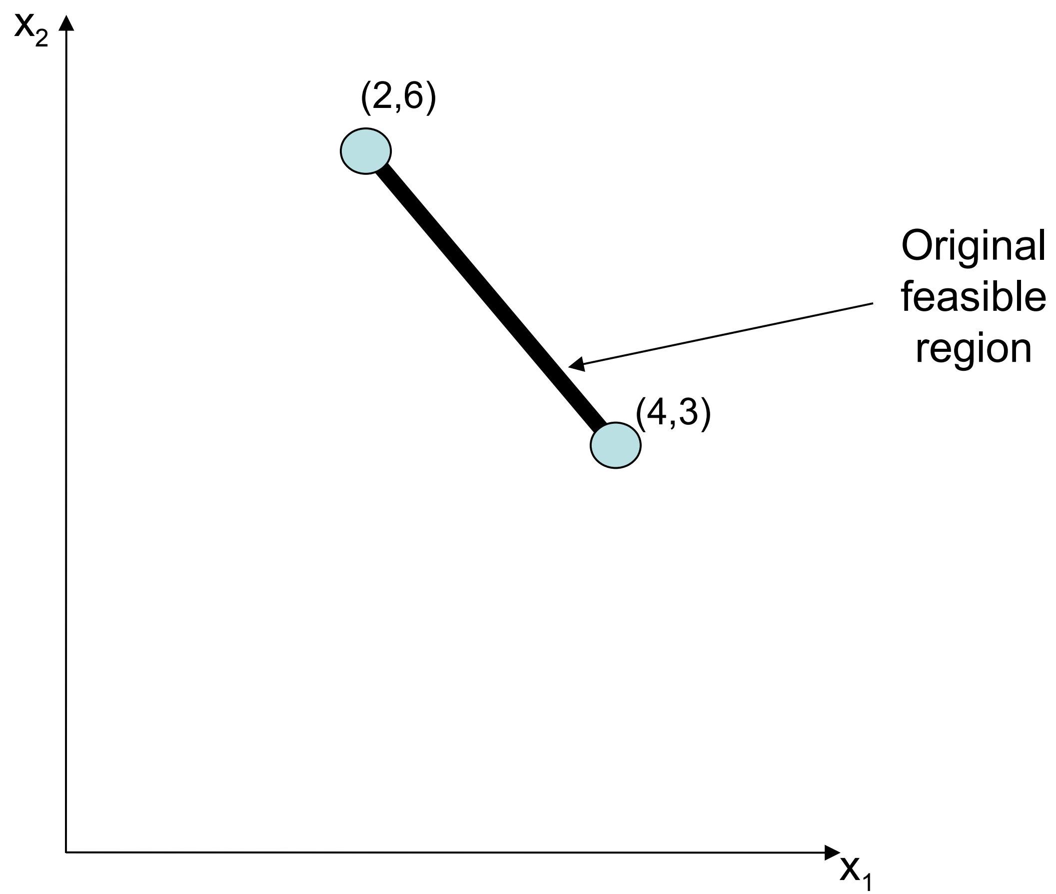
Basic variable	Z	x_1	x_2	r.h.s.

Solving the new problem (2)

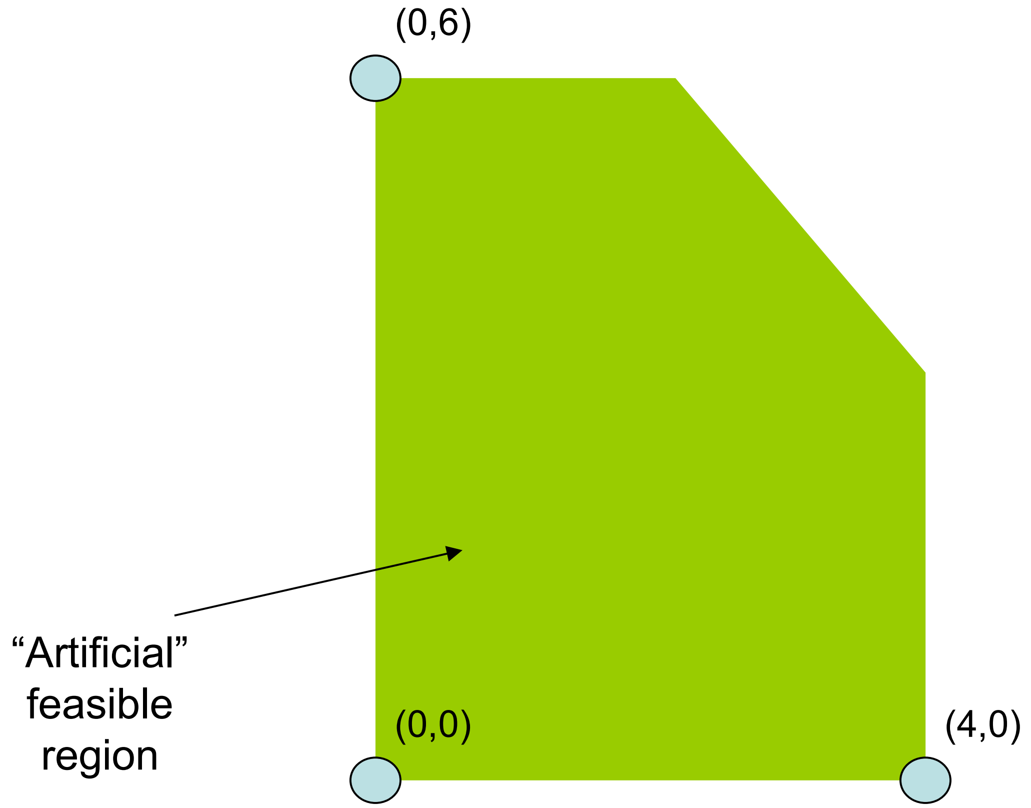
Basic variable	Z	x_1	x_2		r.h.s.

Basic variable	Z	x_1	x_2		r.h.s.

Geometric Insight into Big M



Geometric Insight into Big M



Negative RHS Values

- Consider a constraint with $b_j < 0$
  - e.g. The number of windows produced should be at most 5 less than the number of doors produced:
- Easy fix:
- Tough consequences

$\geq$ Constraints

- Consider the Wyndor Glass problem, where Plant 3 constraint is changed as follows:

$$
\begin{aligned}
\text{Maximize} \quad & Z = 3x_1 + 5x_2 \\
\text{subject to} \quad & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \geq 18 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- Convert to augmented form:

- Then use Big-M method

Variables Allowed to be Negative

$$\text{Maximize} \quad Z = 3x_1 + 5x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 \quad\quad & \leq 4 \\ 2x_2 & \leq 12 \\ 3x_1 + 2x_2 & \leq 18 \end{aligned}$$

$\mathbf{x_1 \geq -10}$, $x_2 \geq 0$

(When there is a bound on the negative values allowed)

Variables Allowed to be Negative

$$
\begin{aligned}
\text{Maximize} \quad Z = \; & 3x_1 + 5x_2 \\
\text{subject to} \quad & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18
\end{aligned}
$$

x_1 unrestricted in sign, $x_2 \geq 0$

(When there is no bound on the negative values allowed)

Traffic Signal Example

- Set green times for a crossroad
- Allocate a given cycle time of 62 seconds to E-W and N-S directions, less an all-red time of 2 seconds
- E-W direction should receive
  - at least 10 seconds of green time
  - at least 5 seconds less than twice the N-S green time
- A linear estimate for the average delay per car per lane is given as 120 - 2 (E-W GT) - 3 (N-S GT)
- Find the green time allocation to minimize average delay