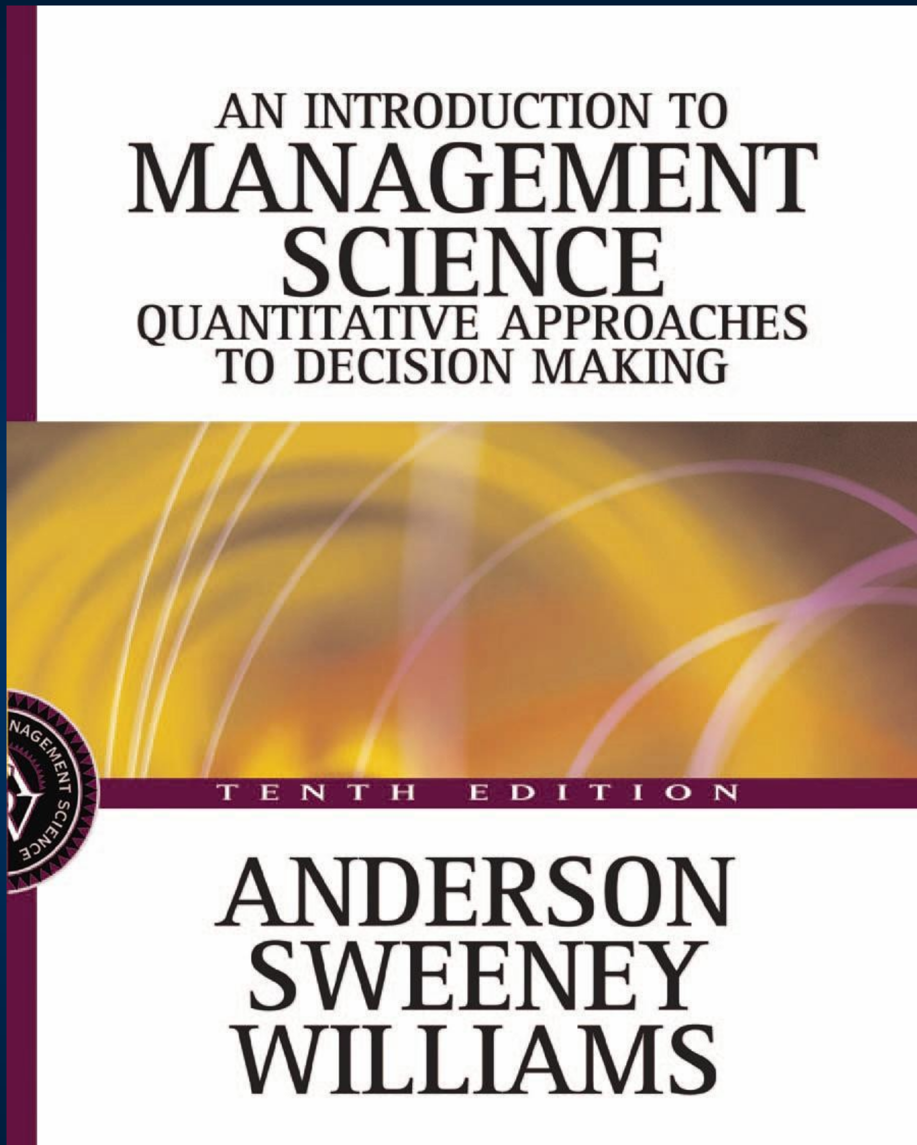

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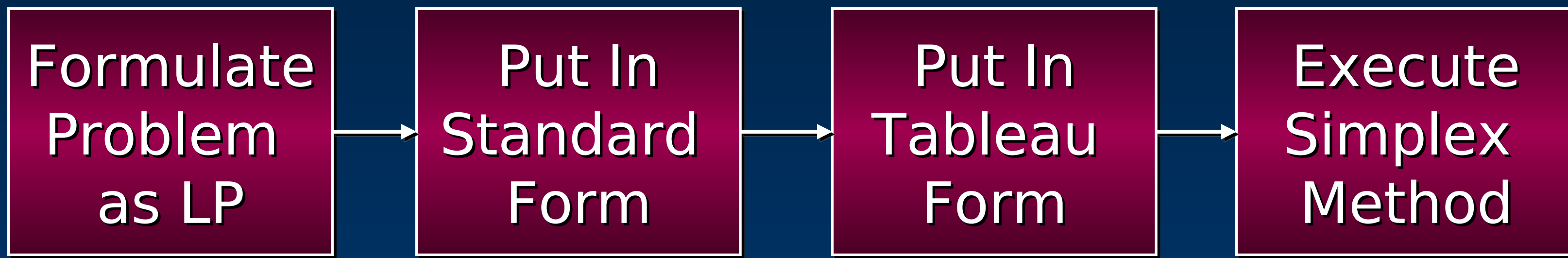
Chapter 5
Linear Programming: The Simplex Method

- An Overview of the Simplex Method
- Standard Form
- Tableau Form
- Setting Up the Initial Simplex Tableau
- Improving the Solution
- Calculating the Next Tableau
- Solving a Minimization Problem
- Special Cases

Overview of the Simplex Method

- Steps Leading to the Simplex Method



Example: Initial Formulation

- A Minimization Problem

$$
\begin{aligned}
\text{MIN} \quad & 2x_1 - 3x_2 - 4x_3 \\
\text{s. t.} \quad & x_1 + x_2 + x_3 \leq 30 \\
& 2x_1 + x_2 + 3x_3 \geq 60 \\
& x_1 - x_2 + 2x_3 = 20 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Standard Form

- An LP is in standard form when:
  - All variables are non-negative
  - All constraints are equalities
- Putting an LP formulation into standard form involves:
  - Adding slack variables to “$\leq$“ constraints
  - Subtracting surplus variables from “$\geq$” constraints.

Example: Standard Form

- Problem in Standard Form

$$\text{MIN} \quad 2x_1 - 3x_2 - 4x_3$$

$$\text{s. t.} \quad x_1 + x_2 + x_3 + s_1 = 30$$

$$2x_1 + x_2 + 3x_3 - s_2 = 60$$

$$x_1 - x_2 + 2x_3 = 20$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Tableau Form

- A set of equations is in tableau form if for each equation:
  - its right hand side (RHS) is non-negative, and
  - there is a basic variable. (A basic variable for an equation is a variable whose coefficient in the equation is +1 and whose coefficient in all other equations of the problem is 0.)
- To generate an initial tableau form:
  - An artificial variable must be added to each constraint that does not have a basic variable.

Example: Tableau Form

- Problem in Tableau Form

$$\text{MIN} \quad 2x_1 - 3x_2 - 4x_3 + 0s_1 - 0s_2 + Ma_2 + Ma_3$$

$$\text{s. t.} \quad x_1 + x_2 + x_3 + s_1 = 30$$

$$2x_1 + x_2 + 3x_3 - s_2 + a_2 = 60$$

$$x_1 - x_2 + 2x_3 + a_3 = 20$$

$$x_1, x_2, x_3, s_1, s_2, a_2, a_3 \geq 0$$

Simplex Tableau

- The simplex tableau is a convenient means for performing the calculations required by the simplex method.

Setting Up Initial Simplex Tableau

- Step 1: If the problem is a minimization problem, multiply the objective function by -1.
- Step 2: If the problem formulation contains any constraints with negative right-hand sides, multiply each constraint by -1.
- Step 3: Add a slack variable to each $\leq$ constraint.
- Step 4: Subtract a surplus variable and add an artificial variable to each $\geq$ constraint.

Setting Up Initial Simplex Tableau

- Step 5: Add an artificial variable to each = constraint.
- Step 6: Set each slack and surplus variable's coefficient in the objective function equal to zero.
- Step 7: Set each artificial variable's coefficient in the objective function equal to -*M*, where *M* is a very large number.
- Step 8: Each slack and artificial variable becomes one of the basic variables in the initial basic feasible solution.

Simplex Method

- Step 1: Determine Entering Variable
  - Identify the variable with the most positive value in the c_j - z_j row. (The entering column is called the pivot column.)
- Step 2: Determine Leaving Variable
  - For each positive number in the entering column, compute the ratio of the right-hand side values divided by these entering column values.
  - If there are no positive values in the entering column, STOP; the problem is unbounded.
  - Otherwise, select the variable with the minimal ratio. (The leaving row is called the pivot row.)

Simplex Method

- Step 3: Generate Next Tableau
  - Divide the pivot row by the pivot element (the entry at the intersection of the pivot row and pivot column) to get a new row. We denote this new row as (row *).
  - Replace each non-pivot row *i* with:

    [new row *i*] = [current row *i*] - [(a_{ij}) x (row *)],

    where a_{ij} is the value in entering column *j* of row *i*

Simplex Method

- Step 4: Calculate z_j Row for New Tableau
  - For each column *j*, multiply the objective function coefficients of the basic variables by the corresponding numbers in column *j* and sum them.

Simplex Method

- Step 5: Calculate c_j - z_j Row for New Tableau
  - For each column *j*, subtract the z_j row from the c_j row.
  - If none of the values in the c_j - z_j row are positive, GO TO STEP 1.
  - If there is an artificial variable in the basis with a positive value, the problem is infeasible. STOP.
  - Otherwise, an optimal solution has been found. The current values of the basic variables are optimal. The optimal values of the non-basic variables are all zero.
  - If any non-basic variable's c_j - z_j value is 0, alternate optimal solutions might exist. STOP.

Example: Simplex Method

- Solve the following problem by the simplex method:

$$\text{Max} \quad 12x_1 + 18x_2 + 10x_3$$

$$\text{s.t.} \quad 2x_1 + 3x_2 + 4x_3 \leq 50$$

$$x_1 - x_2 - x_3 \geq 0$$

$$x_2 - 1.5x_3 \geq 0$$

$$x_1, x_2, x_3 \geq 0$$

Example: Simplex Method

- Writing the Problem in Tableau Form

  We can avoid introducing artificial variables to the second and third constraints by multiplying each by -1 (making them $\leq$ constraints). Thus, slack variables s_1, s_2, and s_3 are added to the three constraints.

$$\text{Max} \quad 12x_1 + 18x_2 + 10x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 + 3x_2 + 4x_3 + s_1 \qquad\qquad &= 50 \\ -x_1 + x_2 + x_3 \qquad + s_2 \qquad &= 0 \\ -x_2 + 1.5x_3 \qquad\qquad + s_3 &= 0 \end{aligned}$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Example: Simplex Method

- Initial Simplex Tableau

Basis	c_B	x_1	x_2	x_3	s_1	s_2	s_3	
		12	18	10	0	0	0	
s_1	0	2	3	4	1	0	0	50
s_2	0	-1	1	1	0	1	0	0
(* row)								
s_3	0	0	-1	1.5	0	0	1	0
z_j		0	0	0	0	0	0	0
$c_j - z_j$		12	18	10	0	0	0	

Example: Simplex Method

- Iteration 1
  - Step 1: Determine the Entering Variable

    The most positive $c_j - z_j = 18$. Thus x_2 is the entering variable.

  - Step 2: Determine the Leaving Variable

    Take the ratio between the right hand side and positive numbers in the x_2 column:

    50/3 = 16 2/3

    0/1 = 0 ← minimum

    s_2 is the leaving variable and the 1 is the pivot element.

Example: Simplex Method

- Iteration 1 (continued)
  - Step 3: Generate New Tableau

    Divide the second row by 1, the pivot element. Call the "new" (in this case, unchanged) row the "* row".

    Subtract 3 x (* row) from row 1.

    Subtract -1 x (* row) from row 3.

    New rows 1, 2, and 3 are shown in the upcoming tableau.

Example: Simplex Method

- Iteration 1 (continued)
  - Step 4: Calculate z_j Row for New Tableau

    The new z_j row values are obtained by multiplying the c_B column by each column, element by element and summing.

    For example, $z_1 = 5(0) + -1(18) + -1(0) = -18$.

Example: Simplex Method

- Iteration 1 (continued)
  - Step 5: Calculate c_j - z_j Row for New Tableau

The new c_j-z_j row values are obtained by subtracting z_j value in a column from the c_j value in the same column.

For example, c_1-z_1 = 12 - (-18) = 30.

Example: Simplex Method

- Iteration 1 (continued) - New Tableau

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	12	18	10	0	0	0	
s_1 (* row)	0	5	0	1	1	-3	0	50
x_2	18	-1	1	1	0	1	0	0
s_3	0	-1	0	2.5	0	1	1	0
z_j		-18	18	18	0	18	0	0
$c_j - z_j$		30	0	-8	0	-18	0	

Example: Simplex Method

- Iteration 2
  - Step 1: Determine the Entering Variable

    The most positive c_j - z_j = 30. x_1 is the entering variable.
  - Step 2: Determine the Leaving Variable

Take the ratio between the right hand side and positive numbers in the x_1 column:

10/5 = 2 ← minimum

There are no ratios for the second and third rows because their column elements (-1) are negative.

Thus, s_1 (corresponding to row 1) is the leaving variable and 5 is the pivot element.

Example: Simplex Method

- Iteration 2 (continued)
  - Step 3: Generate New Tableau

    Divide row 1 by 5, the pivot element. (Call this new row 1 the "* row").

    Subtract (-1) x (* row) from the second row.

    Subtract (-1) x (* row) from the third row.
  - Step 4: Calculate z_j Row for New Tableau

The new z_j row values are obtained by multiplying the c_B column by each column, element by element and summing.

For example, $z_3 = .2(12) + 1.2(18) + .2(0) = 24$.

Example: Simplex Method

- Iteration 2 (continued)
  - Step 5: Calculate c_j - z_j Row for New Tableau

The new c_j-z_j row values are obtained by subtracting z_j value in a column from the c_j value in the same column.

For example, c_3-z_3 = 10 - (24) = -14.

Since there are no positive numbers in the c_j - z_j row, this tableau is optimal. The optimal solution is: x_1 = 10; x_2 = 10; x_3 = 0; s_1 = 0; s_2 = 0 s_3 = 10, and the optimal value of the objective function is 300.

Example: Simplex Method

- Iteration 2 (continued) – Final Tableau

Basis	c_B	x_1	x_2	x_3	s_1	s_2	s_3	
		12	18	10	0	0	0	
x_1	12	1	0	.2	.2	-.6	0	10
x_2	18	0	1	1.2	.2	.4	0	10
s_3	0	0	0	2.7	.2	.4	1	10
z_j		12	18	24	6	0	0	300
$c_j - z_j$		0	0	-14	-6	0	0	

(* row)

Special Cases

- Infeasibility
- Unboundedness
- Alternative Optimal Solution
- Degeneracy

Infeasibility

- Infeasibility is detected in the simplex method when an artificial variable remains positive in the final tableau.

Example: Infeasibility

- LP Formulation

$$\text{MAX} \quad 2x_1 + 6x_2$$

$$\text{s. t.} \quad 4x_1 + 3x_2 \leq 12$$

$$2x_1 + x_2 \geq 8$$

$$x_1, x_2 \geq 0$$

Example: Infeasibility

- Final Tableau

		x_1	x_2	s_1	s_2	a_2	
Basis	C_B	2	6	0	0	$-M$	
x_1	2	1	3/4	1/4	0	0	3
a_2	$-M$	0	-1/2	-1/2	-1	1	2
z_j	2	$(1/2)M$	$(1/2)M$	M	$-M$	$-2M$	
	+3/2	+1/2	+6				
$c_j - z_j$	0	$-(1/2)M$	$-(1/2)M$	$-M$	0		
	+9/2	-1/2					

Example: Infeasibility

In the previous slide we see that the tableau is the final tableau because all c_j - $z_j \leq 0$. However, an artificial variable is still positive, so the problem is infeasible.

Unboundedness

- A linear program has an unbounded solution if all entries in an entering column are non-positive.

Example: Unboundedness

- LP Formulation

$$\text{MAX} \quad 2x_1 + 6x_2$$

$$\text{s. t.} \quad 4x_1 + 3x_2 \geq 12$$

$$2x_1 + x_2 \geq 8$$

$$x_1, x_2 \geq 0$$

Example: Unboundedness

- Final Tableau

Basis	c_B	x_1	x_2	s_1	s_2	
		3	4	0	0	
x_2	4	3	1	0	-1	8
s_1	0	2	0	1	-1	3
	z_j	12	4	0	-4	32
	$c_j - z_j$	-9	0	0	4	

Example: Unboundedness

In the previous slide we see that c_4 - z_4 = 4 (is positive), but its column is all non-positive. This indicates that the problem is unbounded.

Alternative Optimal Solution

- A linear program has <u>alternate optimal solutions</u> if the final tableau has a c_j - z_j value equal to 0 for a non-basic variable.

Example: Alternative Optimal Solution

- Final Tableau

Basis	c_B	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
		2	4	6	0	0	0	0	
s_3	0	0	0	2	4	-2	1	0	8
x_2	4	0	1	2	2	-1	0	0	6
x_1	2	1	0	-1	1	2	0	0	4
s_4	0	0	0	1	3	2	0	1	12
z_j		2	4	6	10	0	0	0	32
$c_j - z_j$		0	0	0	-10	0	0	0	

Example: Alternative Optimal Solution

In the previous slide we see that the optimal solution is:

$x_1 = 4$, $x_2 = 6$, $x_3 = 0$, and $z = 32$

Note that x_3 is non-basic and its $c_3 - z_3 = 0$. This 0 indicates that if x_3 were increased, the value of the objective function would not change.

Another optimal solution can be found by choosing x_3 as the entering variable and performing one iteration of the simplex method. The new tableau on the next slide shows an alternative optimal solution is:

$x_1 = 7$, $x_2 = 0$, $x_3 = 3$, and $z = 32$

Example: Alternative Optimal Solution

- New Tableau

Basis	c_B	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
		2	4	6	0	0	0	0	
s_3	0	0	-1	0	2	-1	1	0	2
x_3	6	0	.5	1	1	- .5	0	0	3
x_1	2	1	.5	0	2	1.5	0	0	7
s_4	0	0	- .5	0	2	2.5	0	1	9
z_j		2	4	6	10	0	0	0	32
$c_j - z_j$		0	0	0	-10	0	0	0	

Degeneracy

- A degenerate solution to a linear program is one in which at least one of the basic variables equals 0.
- This can occur at formulation or if there is a tie for the minimizing value in the ratio test to determine the leaving variable.
- When degeneracy occurs, an optimal solution may have been attained even though some $c_j - z_j > 0$.
- Thus, the condition that $c_j - z_j \leq 0$ is sufficient for optimality, but not necessary.

End of Chapter 5