

AN INTRODUCTION TO
**MANAGEMENT
SCIENCE**
QUANTITATIVE APPROACHES
TO DECISION MAKING

TENTH EDITION

**ANDERSON
SWEENEY
WILLIAMS**

Slides Prepared by

JOHN S. LOUCKS

St. Edward's University

Chapter 6

Simplex-Based Sensitivity Analysis and Duality

- Sensitivity Analysis with the Simplex Tableau
- Duality

Objective Function Coefficients and Range of Optimality

- The range of optimality for an objective function coefficient is the range of that coefficient for which the current optimal solution will remain optimal (keeping all other coefficients constant).
- The objective function value might change is this range.

Objective Function Coefficients and Range of Optimality

- Given an optimal tableau, the range of optimality for c_k can be calculated as follows:
 - Change the objective function coefficient to c_k in the c_j row.
 - If x_k is basic, then also change the objective function coefficient to c_k in the c_B column and recalculate the z_j row in terms of c_k .
 - Recalculate the $c_j - z_j$ row in terms of c_k . Determine the range of values for c_k that keep all entries in the $c_j - z_j$ row less than or equal to 0.

Objective Function Coefficients and Range of Optimality

- If c_k changes to values outside the range of optimality, a new $c_j - z_j$ row may be generated. The simplex method may then be continued to determine a new optimal solution.

Shadow Price

- A shadow price for a constraint is the increase in the objective function value resulting from a one unit increase in its right-hand side value.
- Shadow prices and dual prices on *The Management Scientist* output are the same thing for maximization problems and negative of each other for minimization problems.

Shadow Price

- Shadow prices are found in the optimal tableau as follows:
 - "less than or equal to" constraint -- z_j value of the corresponding slack variable for the constraint
 - "greater than or equal to" constraint -- negative of the z_j value of the corresponding surplus variable for the constraint
 - "equal to" constraint -- z_j value of the corresponding artificial variable for the constraint.

Right-Hand Side Values and Range of Feasibility

- The range of feasibility for a right hand side coefficient is the range of that coefficient for which the shadow price remains unchanged.
- The range of feasibility is also the range for which the current set of basic variables remains the optimal set of basic variables (although their values change.)

Right-Hand Side Values and Range of Feasibility

- The range of feasibility for a right-hand side coefficient of a "less than or equal to" constraint, b_k , is calculated as follows:
 - Express the right-hand side in terms of Δb_k by adding Δb_k times the column of the k -th slack variable to the current optimal right hand side.
 - Determine the range of Δb_k that keeps the right-hand side greater than or equal to 0.
 - Add the original right-hand side value b_k (from the original tableau) to these limits for Δb_k to determine the range of feasibility for b_k .

Right-Hand Side Values and Range of Feasibility

- The range of feasibility for "greater than or equal to" constraints is similarly found except one subtracts Δb_k times the current column of the k -th surplus variable from the current right hand side.
- For equality constraints this range is similarly found by adding Δb_k times the current column of the k -th artificial variable to the current right hand side. Otherwise the procedure is the same.

Simultaneous Changes

- For simultaneous changes of two or more objective function coefficients the 100% rule provides a guide to whether the optimal solution changes.
- It states that as long as the sum of the percent changes in the coefficients from their current value to their maximum allowable increase or decrease does not exceed 100%, the solution will not change.
- Similarly, for shadow prices, the 100% rule can be applied to changes in the the right hand side coefficients.

Canonical Form

- A maximization linear program is said to be in canonical form if all constraints are "less than or equal to" constraints and the variables are non-negative.
- A minimization linear program is said to be in canonical form if all constraints are "greater than or equal to" constraints and the variables are non-negative.

Canonical Form

- Convert any linear program to a maximization problem in canonical form as follows:
 - minimization objective function:
 - multiply it by -1
 - "less than or equal to" constraint:
 - leave it alone
 - "greater than or equal to" constraint:
 - multiply it by -1
 - "equal to" constraint:
 - form two constraints, one "less than or equal to", the other "greater or equal to"; then multiply this "greater than or equal to" constraint by -1.

Primal and Dual Problems

- Every linear program (called the primal) has associated with it another linear program called the dual.
- The dual of a maximization problem in canonical form is a minimization problem in canonical form.
- The rows and columns of the two programs are interchanged and hence the objective function coefficients of one are the right hand side values of the other and vice versa.

Primal and Dual Problems

- The optimal value of the objective function of the primal problem equals the optimal value of the objective function of the dual problem.
- Solving the dual might be computationally more efficient when the primal has numerous constraints and few variables.

Primal and Dual Variables

- The dual variables are the "value per unit" of the corresponding primal resource, i.e. the shadow prices. Thus, they are found in the z_j row of the optimal simplex tableau.
- If the dual is solved, the optimal primal solution is found in z_j row of the corresponding surplus variable in the optimal dual tableau.
- The optimal value of the primal's slack variables are the negative of the $c_j - z_j$ entries in the optimal dual tableau for the dual variables.

Example: Jonni's Toy Co.

Jonni's Toy Co. produces stuffed toy animals and is gearing up for the Christmas rush by hiring temporary workers giving it a total production crew of 30 workers. Jonni's makes two sizes of stuffed animals. The profit, the production time and the material used per toy animal is summarized on the next slide. Workers work 8 hours per day and there are up to 2000 pounds of material available daily.

What is the optimal daily production mix?

Example: Jonni's Toy Co.

Used	Toy	Unit	Production	Material
(lbs.)	<u>Size</u>	<u>Profit</u>	<u>Time (hrs.)</u>	<u>Per Unit</u>
1	Small	\$3	.10	
2	Large	\$8	.30	

Example: Jonni's Toy Co.

■ LP Formulation

x_1 = number of small stuffed animals produced daily

x_2 = number of large stuffed animals produced daily

$$\begin{array}{ll} \text{Max} & 3x_1 + 8x_2 \\ \text{s.t.} & .1x_1 + .3x_2 \leq 240 \\ & x_1 + 2x_2 \leq 2000 \\ & x_1, x_2 \geq 0 \end{array}$$

Example: Jonni's Toy Co.

■ Simplex Method: First Tableau

		x_1	x_2	s_1	s_2	
Basis	c_B	3	8	0	0	
	s_1	0	.1	.3	1	0
	s_2	0	1	2	0	1
	z_j	0	0	0	0	0
$c_j - z_j$		3	8	0	0	

240

2000

0

Example: Jonni's Toy Co.

■ Simplex Method: Second Tableau

		x_1	x_2	s_1	s_2	
Basis	c_B	3	8	0	0	
x_2	8	1/3	1	10/3	0	800
s_2	0	1/3	0	-20/3	1	400
	z_j	8/3	8	80/3	0	
	$c_j - z_j$	1/3	0	-80/3	0	

6400

Example: Jonni's Toy Co.

■ Simplex Method: Third Tableau

Basis c_B		x_1	x_2	s_1	s_2		
		3	8	0	0		
x_2	8	0	1	10	-1		400
x_1	3	1	0	-20	3		1200
z_j		3	8	20	1		6800
$c_j - z_j$		0	0	-20	-1		

Example: Jonni's Toy Co.

■ Optimal Solution

- Question:

How many animals of each size should be produced daily and what is the resulting daily profit?

- Answer:

Produce 1200 small animals and 400 large animals daily for a total profit of \$6,800.

Example: Jonni's Toy Co.

- Range of Optimality for c_1 (small animals)

Replace 3 by c_1 in the objective function row and c_B column. Then recalculate z_j and $c_j - z_j$ rows.

	z_j	c_1	8	80	$-20c_1$	-8	$+3c_1$	3200	+
$1200c_1$									
$c_j - z_j$		0	0	-80	$+20c_1$	8	$-3c_1$		

For the $c_j - z_j$ row to remain non-positive, $8/3 \leq c_1 \leq 4$

Example: Jonni's Toy Co.

- Range of Optimality for c_2 (large animals)

Replace 8 by c_2 in the objective function row and c_B column. Then recalculate z_j and $c_j - z_j$ rows.

	z_j	3	c_2	-60	$+10c_2$	9	$-c_2$	3600
$+ 400c_2$								
$c_j - z_j$		0	0	60	$-10c_2$	-9	$+c_2$	

For the $c_j - z_j$ row to remain non-positive, $6 \leq c_2 \leq 9$

Example: Jonni's Toy Co.

■ Range of Optimality

- **Question:** Will the solution change if the profit on small animals is increased by \$.75? Will the objective function value change?
- **Answer:** If the profit on small stuffed animals is changed to \$3.75, this is within the range of optimality and the optimal solution will not change. However, since x_1 is a basic variable at positive value, changing its objective function coefficient will change the value of the objective function to $3200 + 1200(3.75) = 7700$.

Example: Jonni's Toy Co.

■ Range of Optimality

- **Question:** Will the solution change if the profit on large animals is increased by \$.75? Will the objective function value change?
- **Answer:** If the profit on large stuffed animals is changed to \$8.75, this is within the range of optimality and the optimal solution will not change. However, since x_2 is a basic variable at positive value, changing its objective function coefficient will change the value of the objective function to $3600 + 400(8.75) = 7100$.

Example: Jonni's Toy Co.

■ Range of Optimality and 100% Rule

- **Question:** Will the solution change if the profits on both large and small animals are increased by \$.75? Will the value of the objective function change?
- **Answer:** If both the profits change by \$.75, since the maximum increase for c_1 is \$1 (from \$3 to \$4) and the maximum increase in c_2 is \$1 (from \$8 to \$9), the overall sum of the percent changes is $(.75/1) + (.75/1) = 75\% + 75\% = 150\%$. This total is greater than 100%; both the optimal solution and the value of the objective function change.

Example: Jonni's Toy Co.

■ Shadow Price

- **Question:** The unit profits do not include a per unit labor cost. Given this, what is the maximum wage Jonni should pay for overtime?
- **Answer:** Since the unit profits do not include a per unit labor cost, man-hours is a sunk cost. Thus the shadow price for man-hours gives the maximum worth of man-hours (overtime). This is found in the z_j row in the s_1 column (since s_1 is the slack for man-hours) and is \$20.

Example: Prime the Cannons!

■ LP Formulation

$$\text{Max } 2x_1 + x_2 + 3x_3$$

$$\text{s.t. } x_1 + 2x_2 + 3x_3 \leq 15$$

$$3x_1 + 4x_2 + 6x_3 \geq 24$$

$$x_1 + x_2 + x_3 = 10$$

$$x_1, x_2, x_3 \geq 0$$

Example: Prime the Cannons!

■ Primal in Canonical Form

- Constraint (1) is a " \leq " constraint. Leave it alone.
- Constraint (2) is a " \geq " constraint. Multiply it by -1.
- Constraint (3) is an "=" constraint. Rewrite this as two constraints, one a " \leq ", the other a " \geq " constraint. Then multiply the " \geq " constraint by -1.

(result on next slide)

Example: Prime the Cannons!

- Primal in Canonical Form (continued)

$$\begin{array}{ll} \text{Max} & 2x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 2x_2 + 3x_3 \leq 15 \\ & -3x_1 - 4x_2 - 6x_3 \leq -24 \\ & x_1 + x_2 + x_3 \leq 10 \\ & -x_1 - x_2 - x_3 \leq -10 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Example: Prime the Cannons!

■ Dual of the Canonical Primal

- There are four dual variables, U_1, U_2, U_3', U_3'' .
- The objective function coefficients of the dual are the RHS of the primal.
- The RHS of the dual is the objective function coefficients of the primal.
- The rows of the dual are the columns of the primal.

(result on next slide)

Example: Prime the Cannons!

- Dual of the Canonical Primal (continued)

2

$$\begin{array}{l} \text{Min} \quad 15U_1 - 24U_2 + 10U_3' - 10U_3'' \\ \text{s.t.} \quad U_1 - 3U_2 + U_3' - U_3'' \geq \end{array}$$

1

$$2U_1 - 4U_2 + U_3' - U_3'' \geq$$

3

$$3U_1 - 6U_2 + U_3' - U_3'' \geq$$

$$U_1, U_2, U_3', U_3'' \geq 0$$

End of Chapter 6

